Short-range Newtonian coupling to GW test-masses

by

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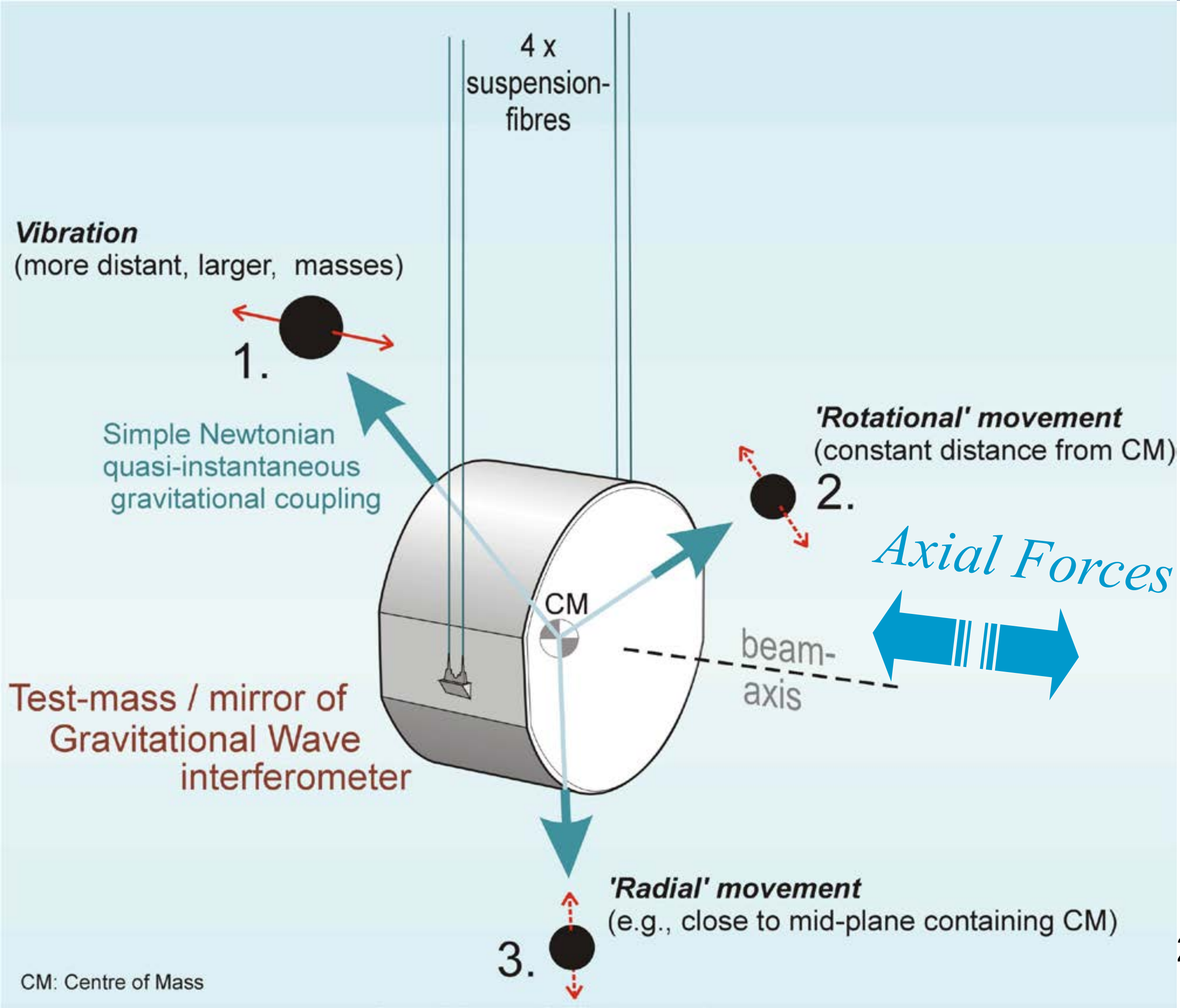
Local (perturbing) gravitational sources

- Suspended test-masses (TMs).
- *Local* mass surrounding TMs may move.
- This movement *will* change Newtonian axial forces on TMs—quasi instantaneously.
- Ostensibly, calculating the effects of such mass movements (1–3) appears obvious and straightforward
  - (are effects negligible ?)



Background—cylindrical test-masses for the Satellite Test of the Equivalence Principle (STEP) experiment

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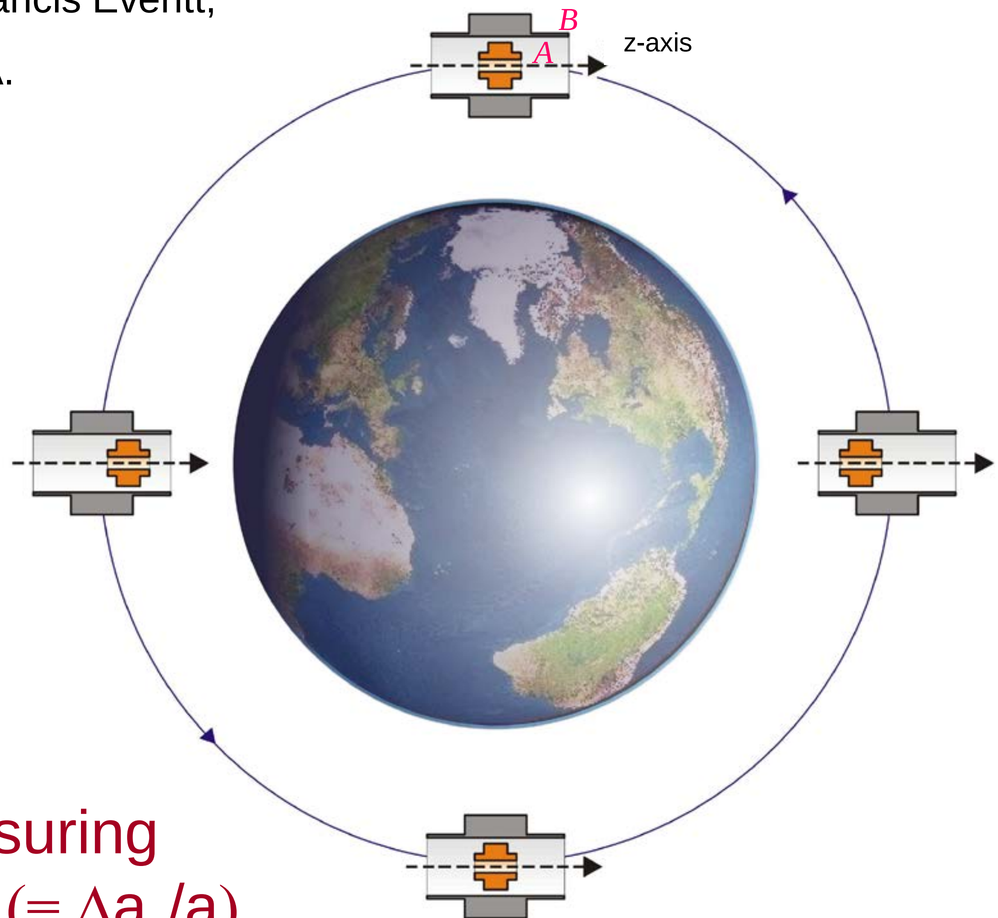
- STEP led by Prof. Francis Everitt, Uni. of Stanford, USA.



- Two bodies *A* and *B*.
- Different materials.
- Permanent free-fall.

- In vacuo.
- 2 K.
- Drag-free spacecraft (as GP-B).

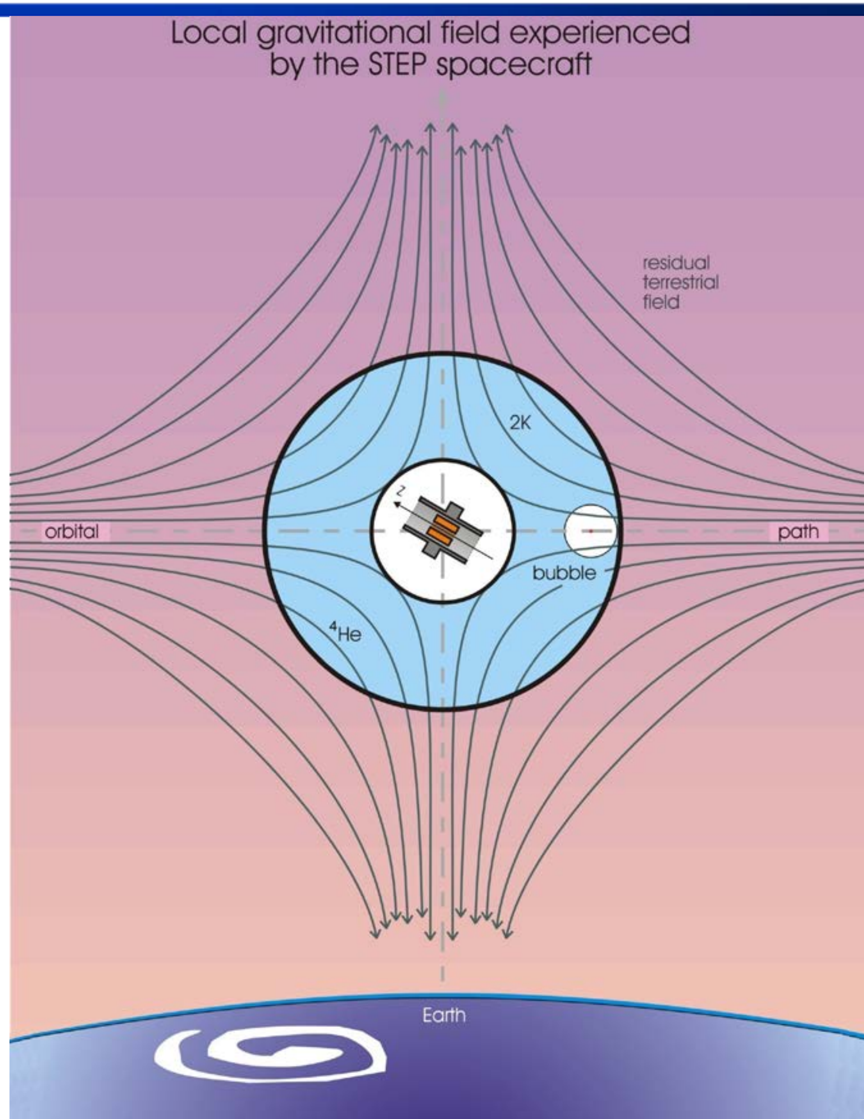
- Earth appears to be orbiting CCW about test-masses.

- Superconducting Linear bearings.
- SQUID differential displacement detection.

- Aimed at measuring Eötvös ratio η ($\equiv \Delta a_z/a$) to 1 part in 10^{18}.

Local gravitational sources for STEP

- Paired cylindrical test-masses
  - Co-axial
  - Concentric.
- Local (potential) *gravitational* sources of systematic error, e.g. bubbles in liquid ^{4}He.
  - A bubble's (negative) mass can pull gravitationally, *and differentially*, on test-masses
  - Synchronous with perceived rotation rate of 'Earth around spacecraft.'
- ➢ Can mimic an EP violating signal.



Newtonian coupling to STEP Test Masses

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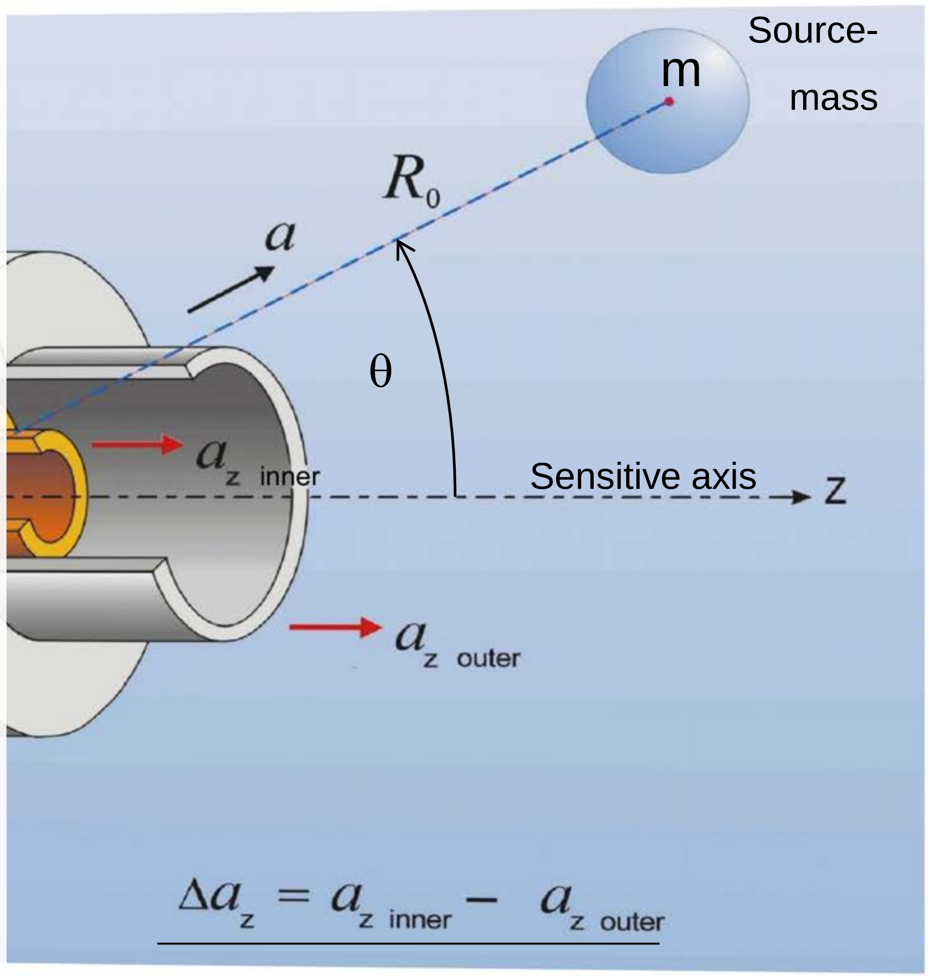
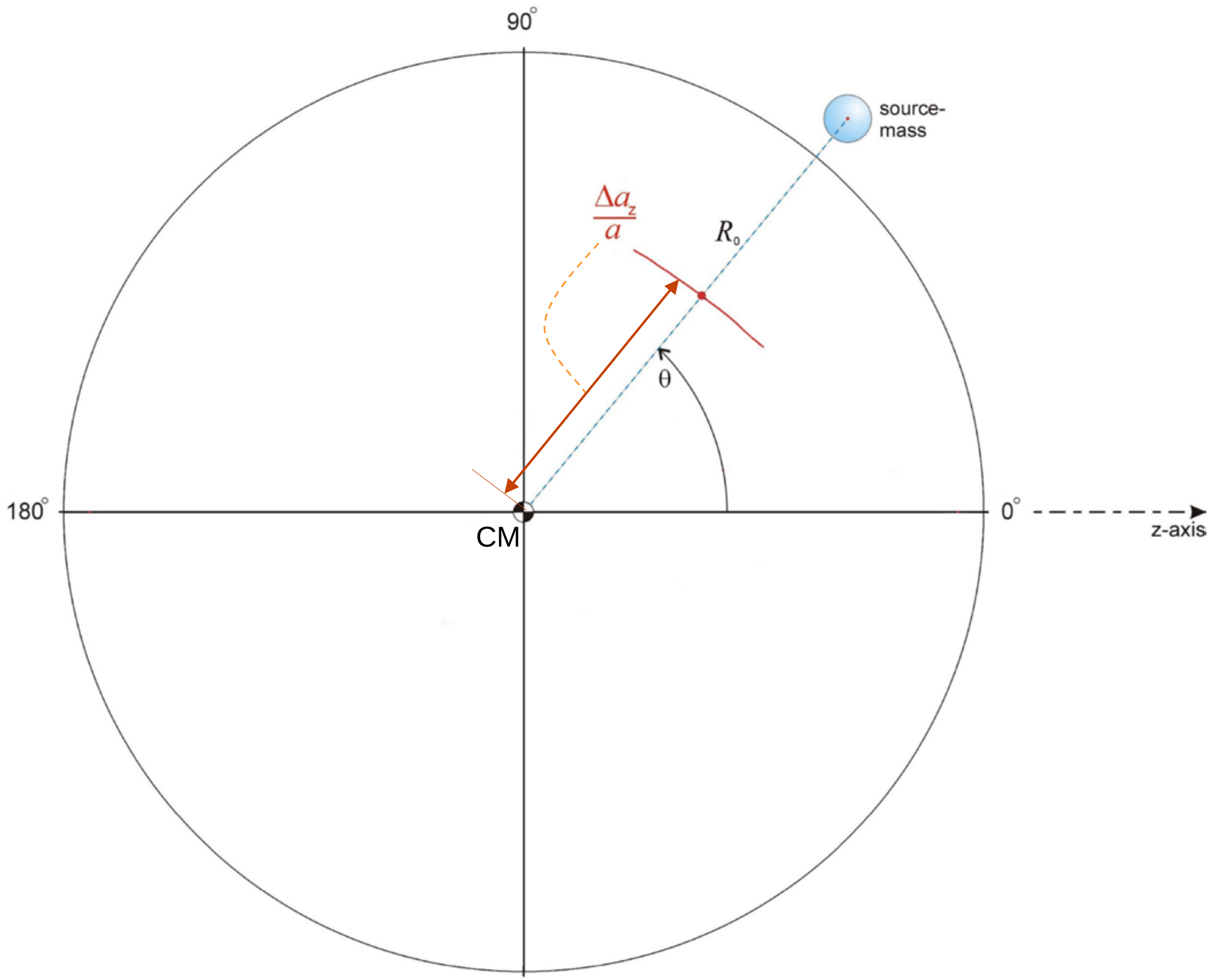
- Radial distance of source mass R_0 was held constant, and θ was stepped in 1º increments through one quadrant.
- At each new source-mass position (value of θ) the total gravitational force on each test mass, due to the point source-mass, was found by integration. From this, the full axial acceleration of each test-mass was deduced.
- The two axial accelerations were differenced to find Δa_z, and the ratio $\Delta a_z/a$ was calculated, where a is the common-mode acceleration.



Cut-away view of test-masses

- Problem resolved itself into finding the axial acceleration of each test-mass due to the local source-mass **m** — here, a perturbing (negative) source.
- ➢ Challenge: determining the best shape for each test-mass so as to minimise Δa_z.

Residual differential acceleration ratio



- Polar plot of ratio $\Delta a_z/a$.

- The 3D integrations of primitive gravitational vector forces took 40 h of computation at that time, but...
  - ...integration over (say) an extended source-mass would involve 6D integrations (3D over source + 3D over each relevant test-mass
    - even today, *Unfeasibly lengthy.*

Fortuitous gravitational balancing…

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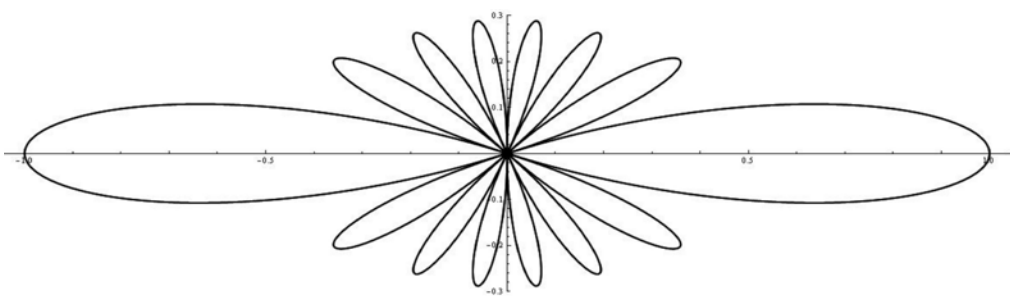
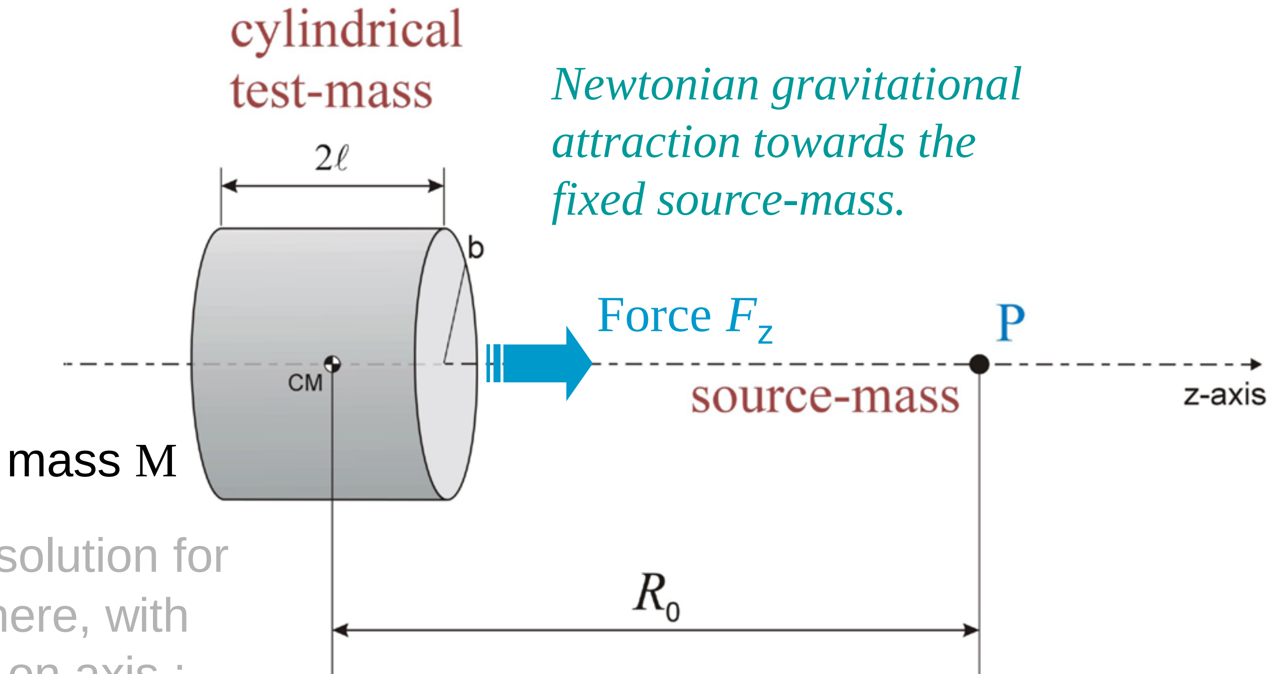
- It turned out in the previous example that gravitational '64-pole' Newtonian coupling dominated the differential acceleration between two almost perfectly balanced test-masses
  - Polar angular appearance was so striking its functional form was perfectly recognizable as
  - the Legendre polynomial $P_7(\cos(\theta))$: -



- Recognition of this relationship led to a far better method for determining the axial acceleration of cylindrical test-masses, due to *point* gravitational sources
  - and so, by superposition, due to *any* gravitational sources.
- The original 40 h calculation by 3D integration was now completed in less than 1 s, and to higher accuracy, using this method.

Single test-mass: source-mass on its cylindrical axis



Closed-form solution for F_z possible here, with source-mass on axis : -

$$F_z = \frac{GM}{b^2\ell}\left(\sqrt{b^2+(R_0-\ell)^2} - \sqrt{b^2+(R_0+\ell)^2} + 2\ell\right).$$

(Unit source mass on z-axis at z = R_0).

becomes

$$F_z = \frac{GM}{R_0^2} \qquad (R_0 >> b, \ell).$$

Newtonian gravitational attraction of an extended body by a (unit) point source-mass

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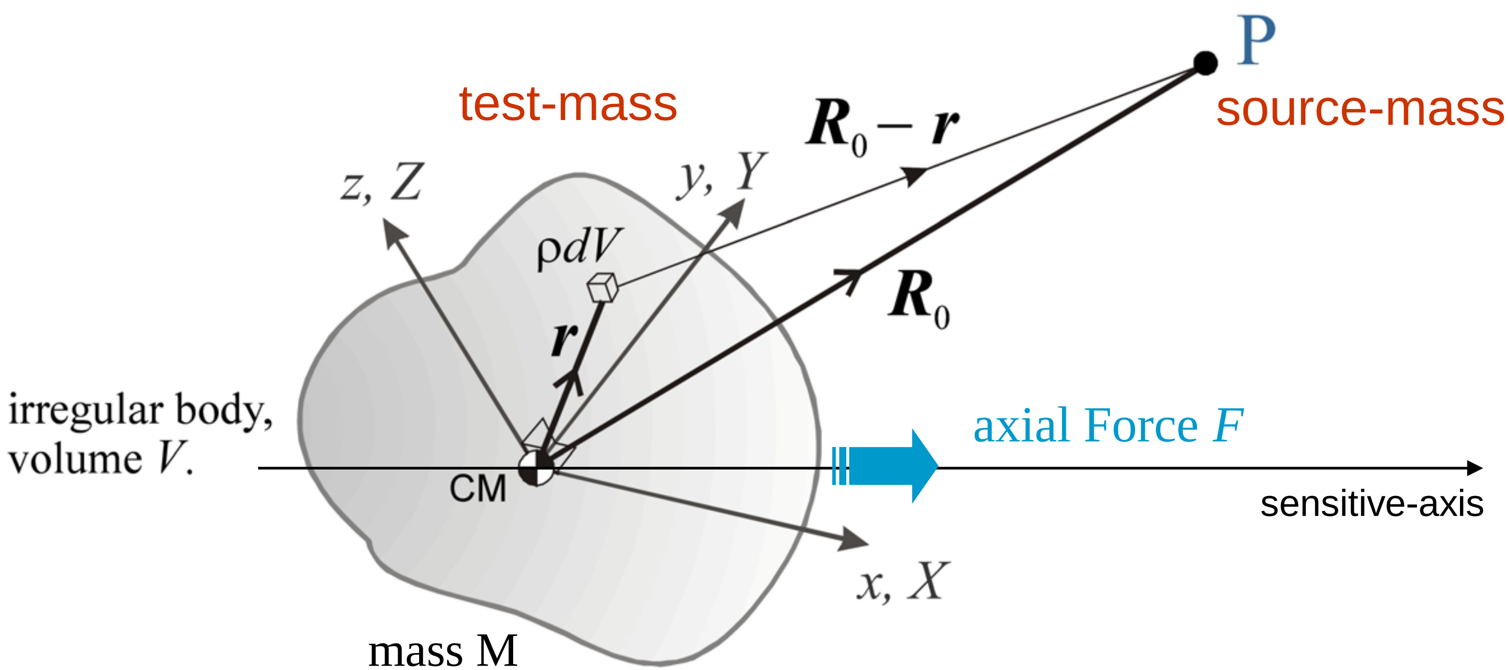


Points within the body described by vector $\boldsymbol{r}$ (x, y, z) .

External field point **P** described by vector $\boldsymbol{R}_0$, (X, Y, Z) .

The density of the body, ρ, may vary throughout its volume, V.

Axes are fixed within body.

Origin is at the CM of the body.

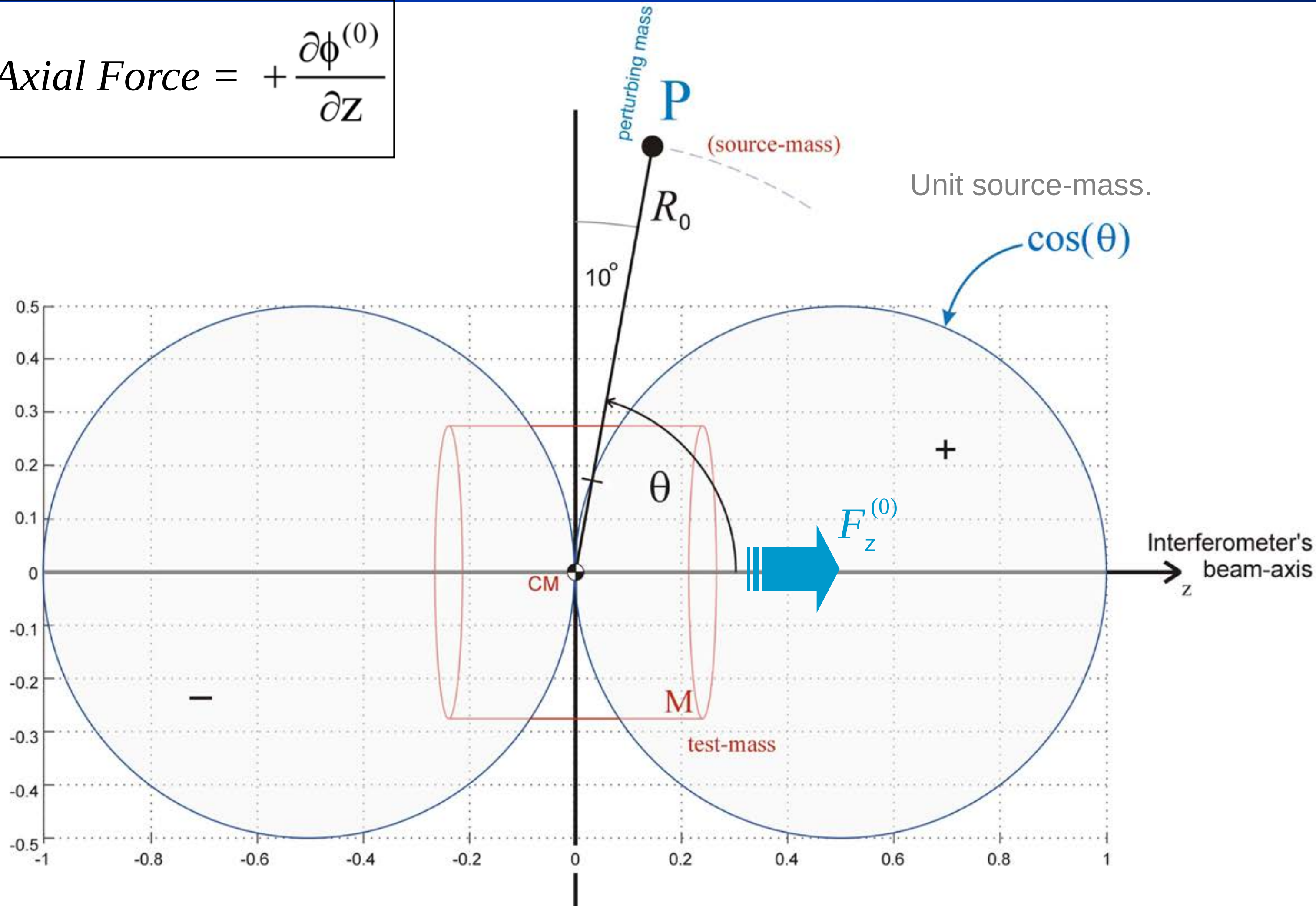
Expansion of the PE about $\boldsymbol{R}_0$

Gravitational PE, source at **P**, is: $\phi = -G \int_V \frac{\rho \, dV}{|\boldsymbol{R}_0 - \boldsymbol{r}|}.$

G is the gravitational constant = 6.67×10^{-11} N.m^2.kg^{-2}; unit source-mass.

Monopolar axial Force, $F_z^{(0)}$

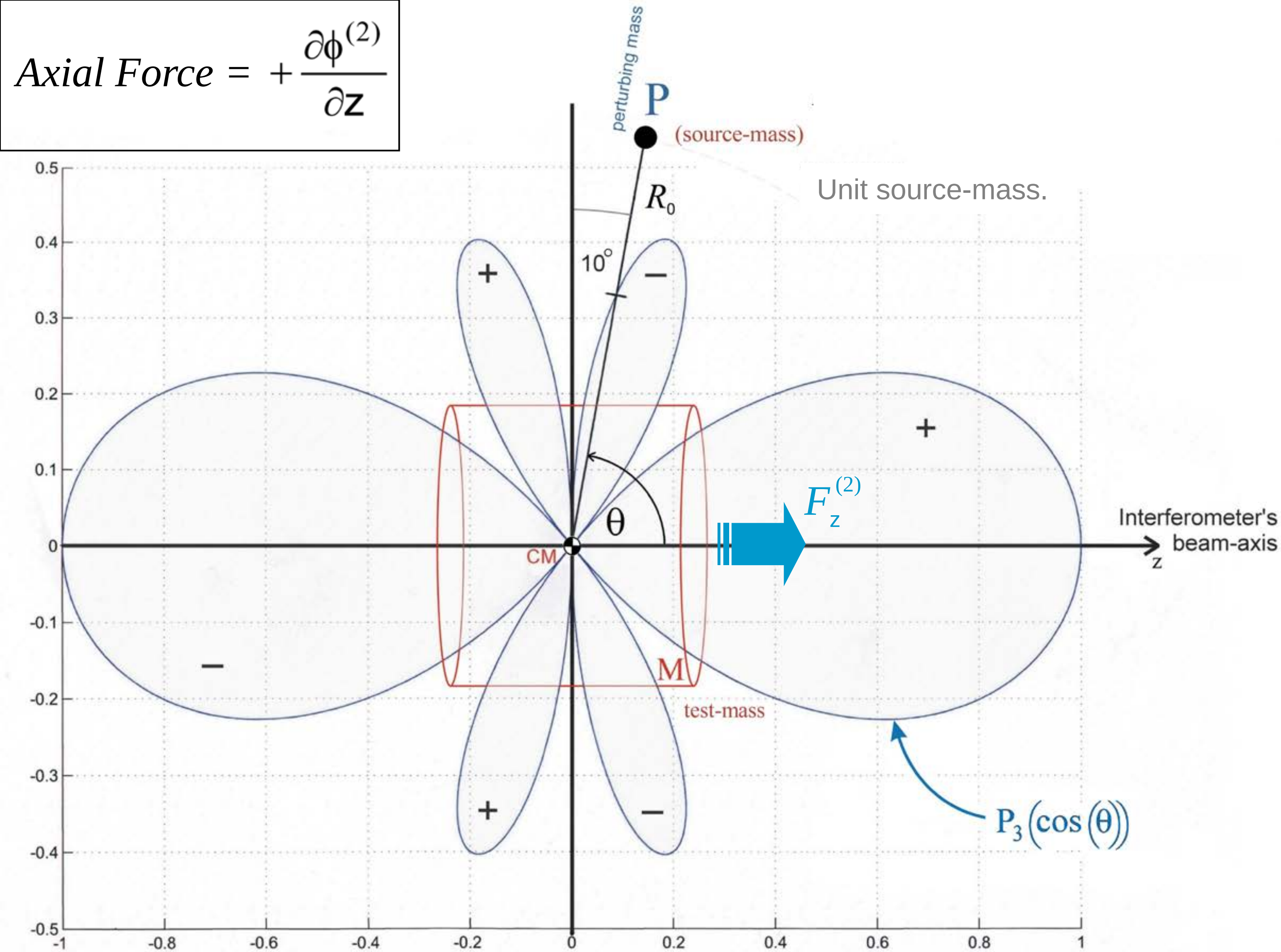
$$\text{Axial Force} = +\frac{\partial\phi^{(0)}}{\partial z}$$



- Arbitrary source-mass position.
- Test-mass shown as 'wire frame' cylinder.
- Origin of polar co-ordinates at its CM.
- R_0 held constant, θ varied.
- Lobe pattern of axial Force *identical* for all cylindrical test-masses—a positive and a negative sphere...
- ...Force always attractive.
- Low axial coupling from source-masses lying to the sides of the test-mass, as shown.

Quadrupolar axial Force, $F_z^{(2)}$

$$\textit{Axial Force} = +\frac{\partial \phi^{(2)}}{\partial z}$$



- Arbitrary source-mass position.

- Test-mass shown as 'wire frame' cylinder.
- R_0 held constant, θ varied.
- Resulting lobe pattern of quadrupolar axial Force is the same for all cylindrical test-masses—but lobes may have opposite signs. It is a figure of revolution about the z-axis (cylindrical symmetry).

➢ Significant axial Force even when perturbing source-mass is almost at right-angles to axis thro' CM of test-mass.

➢ Gravitational quadrupolar Force can be repulsive.

Quadrupolar coupling

The quadrupolar PE can be written as: -

$$\phi^{(2)} = -\frac{G}{6}\sum_{\alpha,\beta}\left[\int_V \rho\left(3x_\alpha x_\beta - r^2\delta_{\alpha\beta}\right)dV\,\frac{\partial^2}{\partial X_\alpha \partial X_\beta}\left(\frac{1}{R_0}\right)\right],$$

where α, β = a, b, c, and

$$\delta_{\alpha\beta} = \begin{cases} 1 & (\alpha = \beta). \\ 0 & (\alpha \neq \beta). \end{cases}$$

- $\boldsymbol{D}_{\alpha\beta} = \int_V \rho\left(3\boldsymbol{x}_\alpha \boldsymbol{x}_\beta - \boldsymbol{r}^2\delta_{\alpha\beta}\right)\boldsymbol{dV}$ are defined to be the nine elements of the 2nd rank symmetric *mass quadrupole tensor*; similarly…

cf. $\boldsymbol{J}_{\alpha\beta} = \int_V \rho\left(\boldsymbol{r}^2\delta_{\alpha\beta} - \boldsymbol{x}_\alpha \boldsymbol{x}_\beta\right)\boldsymbol{dV}$ are the elements of a body's tensor of inertia, [J].

➢ Relative to the body's principal axes of inertia (here, labelled: 1, 2, and 3) **[J]** has only 3 non-zero elements: J_{11} , J_{22} , and J_{33}.

The mass quadrupole tensor

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- Relative to these same *Principal Inertial axes* the gravitational *mass quadrupole tensor* of any body can be written simply in terms of the principal moments of inertia of that body: -

Mean of the other two principal moments of inertia.

$$[\boldsymbol{D}] = -2\begin{bmatrix} J_{11} - \dfrac{J_{22}+J_{33}}{2} & 0 & 0 \\ 0 & J_{22} - \dfrac{J_{11}+J_{33}}{2} & 0 \\ 0 & 0 & J_{33} - \dfrac{J_{11}+J_{22}}{2} \end{bmatrix}$$

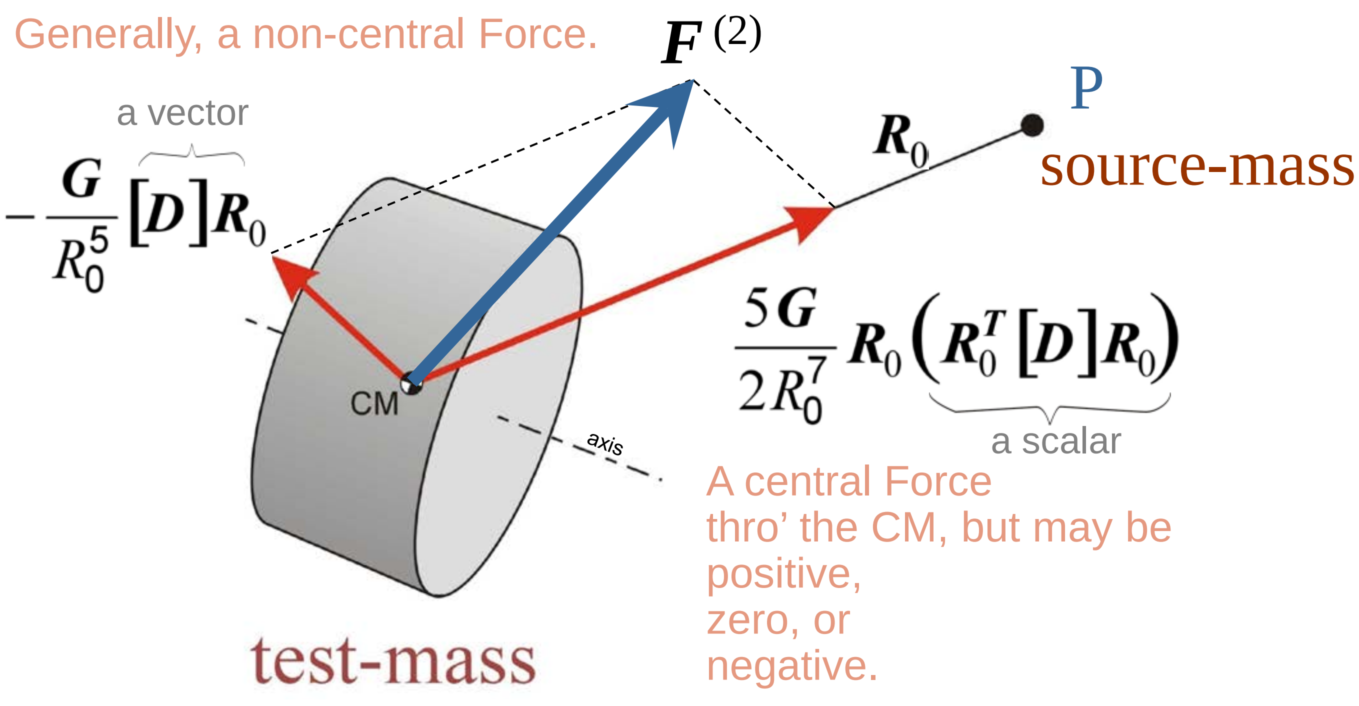
- Therefore, if the body's 'ellipsoid of inertia' is a sphere, then $J_{11} = J_{22} = J_{33}$, and there *can be no quadrupolar gravitational interaction with this body* (*MacCullagh's formula).

*James MacCullach (1809–47) Irish mathematician and physicist.

The quadrupolar Force

(Unit source-mass.)

- Quadrupolar Force:



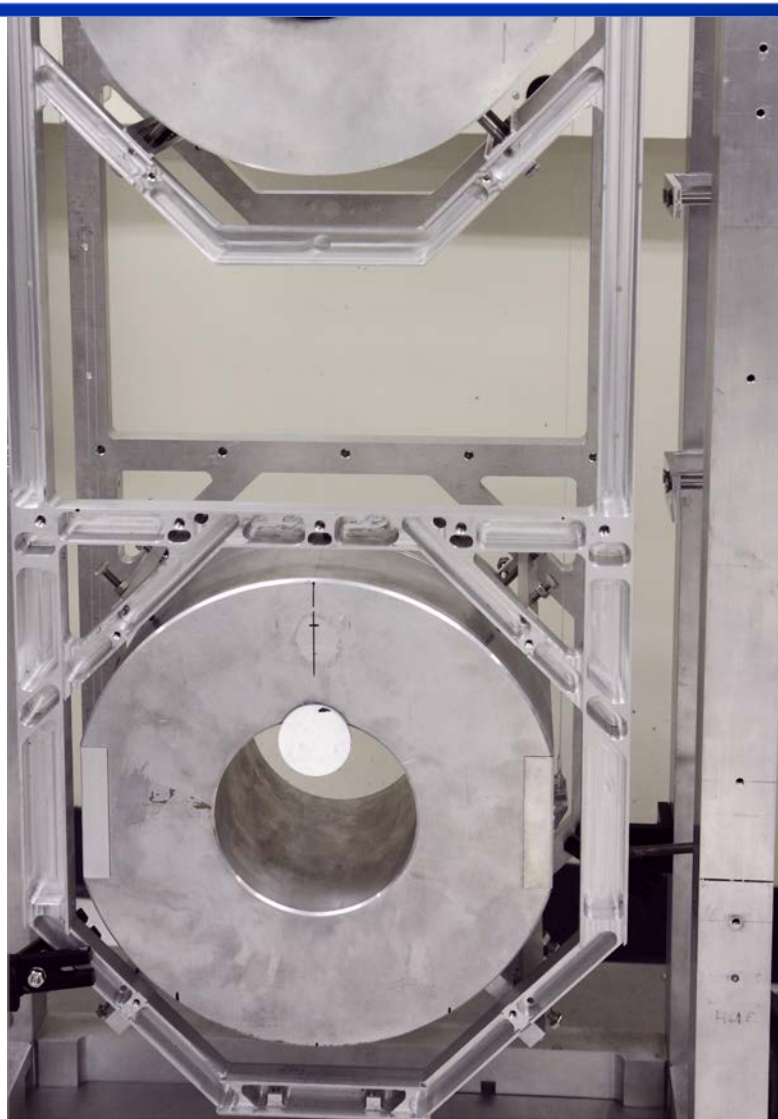
G is the Constant of Gravitation.

$\boldsymbol{R}_0$ is the (column) position vector of the point P measured from the CM of the test-mass, and

$\boldsymbol{R}_0^T$ is the corresponding transpose (row) vector.

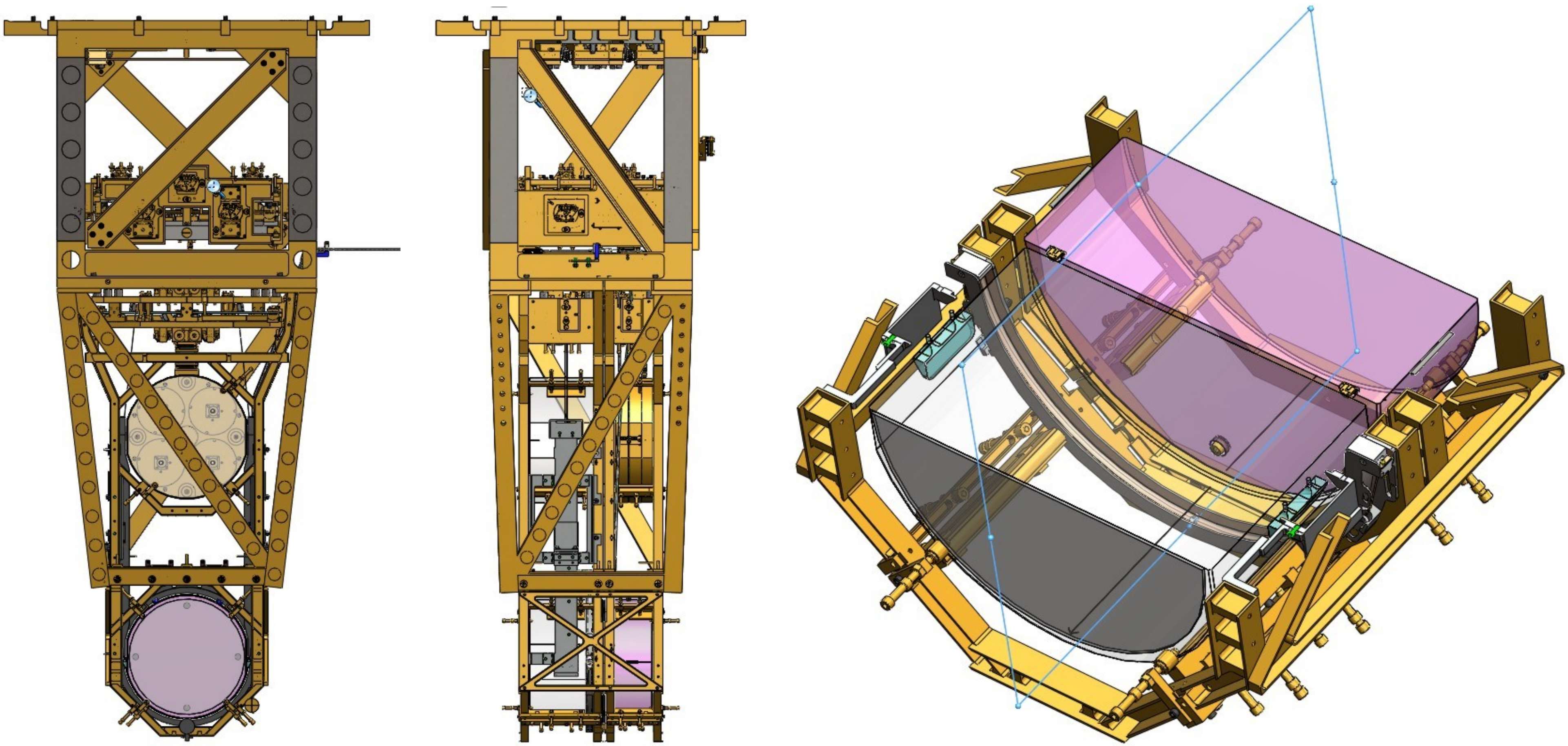
$[\boldsymbol{D}]$ [mass × distance^2] is the ***mass quadrupole moment*** of the test-mass (2nd rank tensor array).

Mass distribution around a suspended test-mass

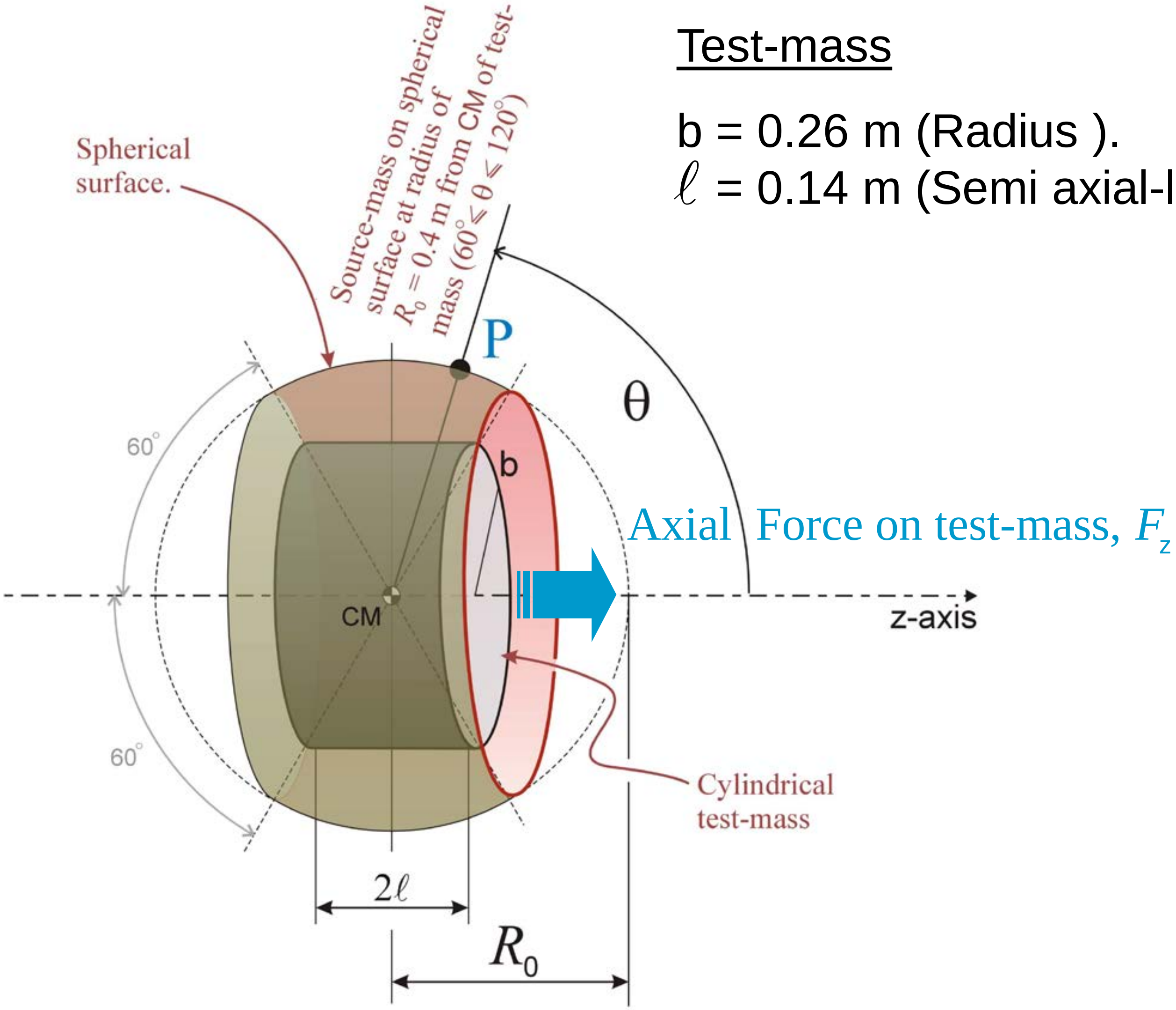


- Test-suspension at MIT (dummy test-mass).
- Glasgow silica-fibre suspension.
- Mechanical structure is necessarily in close proximity to the suspended test-mass
  - Does its mass fall inside the quadrupolar lobes ?
  - What is the relative size of the quadrupolar force ?

Local mass distribution example—the aLIGO suspension



Example: magnitude of ratio $F_z^{(2)}/F_z^{(0)}$



<u>Test-mass</u>

b = 0.26 m (Radius).
ℓ = 0.14 m (Semi axial-length).

- If the axial effect of a point perturbing source-mass is averaged over the spherical surface shown, having radius R_0 = 0.4 m, then
  - The ratio of the magnitude of the quadrupolar to monopolar axial gravitational forces is (in this example) $\boldsymbol{F}_z^{(2)} / \boldsymbol{F}_z^{(0)} = 0.25$; and this ratio $\propto R_0^{-2}$.

Impact of 1 nm Radial or Angular movement of 1 kg source-mass on a_z

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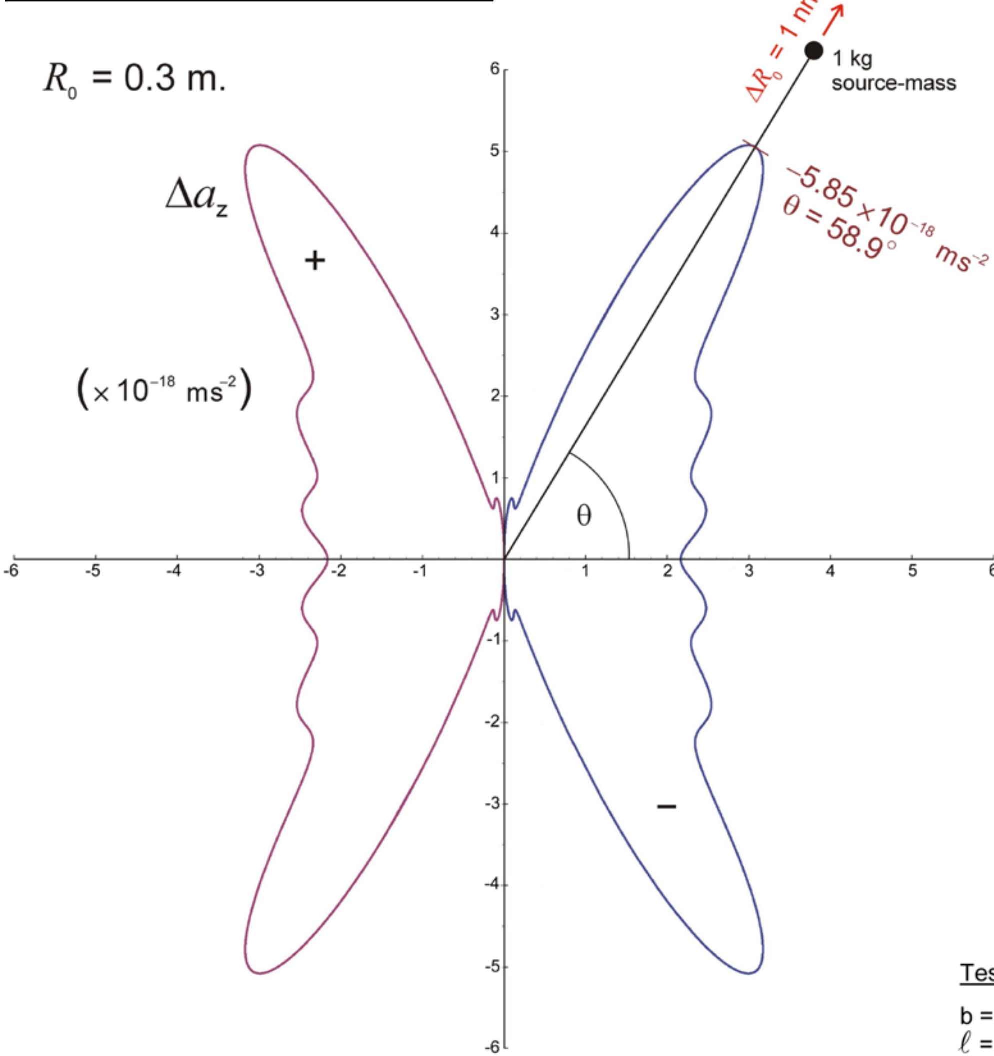
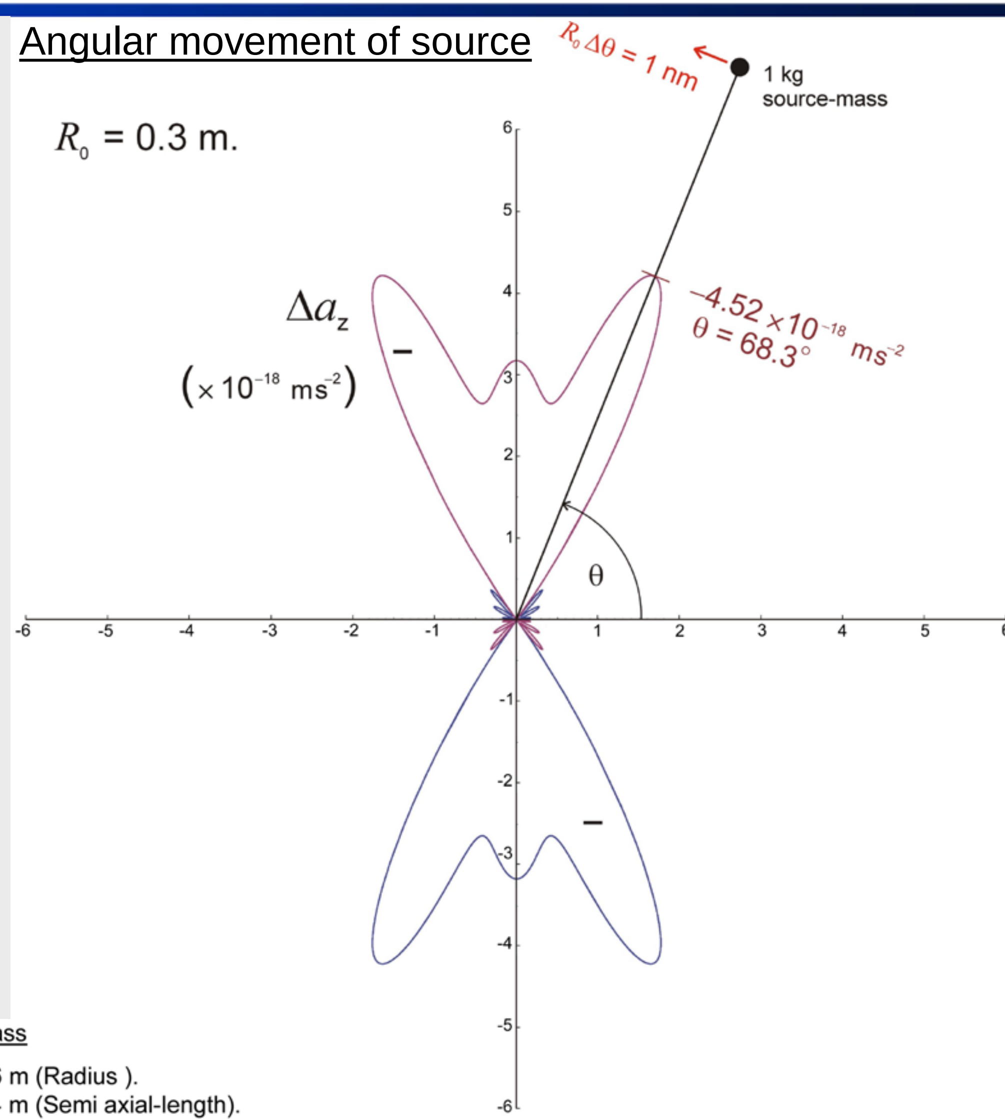
Radial movement of source



Angular movement of source



Test-mass

b = 0.26 m (Radius).

ℓ = 0.14 m (Semi axial-length).

Full moment expansion for test-mass

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$$\boldsymbol{F}_{\mathrm{axial}} = \boldsymbol{G}\,\mathrm{M}\sum_{n=0}^{\infty}\left\{\frac{(2n+1)\,P_{2n+1}(\cos(\theta))}{R_0^{(2n+2)}}\sum_{p=0}^{n}\left(\frac{(-1)^p (2n)!\,\ell^{2[n-p]}\,b^{2p}}{2^{2p}\,p!\,(p+1)!\,(2[n-p]+1)!}\right)\right\}.$$

(Unit source-mass.)

n = 0: Monopole.

n = 1: Quadrupole.

n = 2: Hexadecapole.

n = 3: 64-pole, etc.

.
.
.

➢ Very fast, computationally.

Accuracy of the moment expansion

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Contributions to axial acceleration of cylindrical test-mass for unit source-mass on axis at R_0 = 0.4 m (% of closed-form solution)											
Monopole	Quadrupole	Hexadecapole	64-pole	256-pole	1024-pole	4096-pole	16384-pole	65536-pole	262144-pole	1048576-pole	... (21 terms)
122.31	-23.77	-0.3447	2.810	-1.165	0.06776	0.1803	-0.1021	0.01431	0.01453	-0.01074	...

	Closed-form solution result	Moment expansion result
1 kg source-mass on-axis:	$3.40848 \times 10^{-10}\,\mathrm{ms}^{-2}$.	$3.40848 \times 10^{-10}\,\mathrm{ms}^{-2}$.

Test-mass

b = 0.26 m (Radius).
ℓ = 0.14 m (Semi axial-length).

Background work

Co-workers
Alexey Veryaskin
Xiaohui Xu

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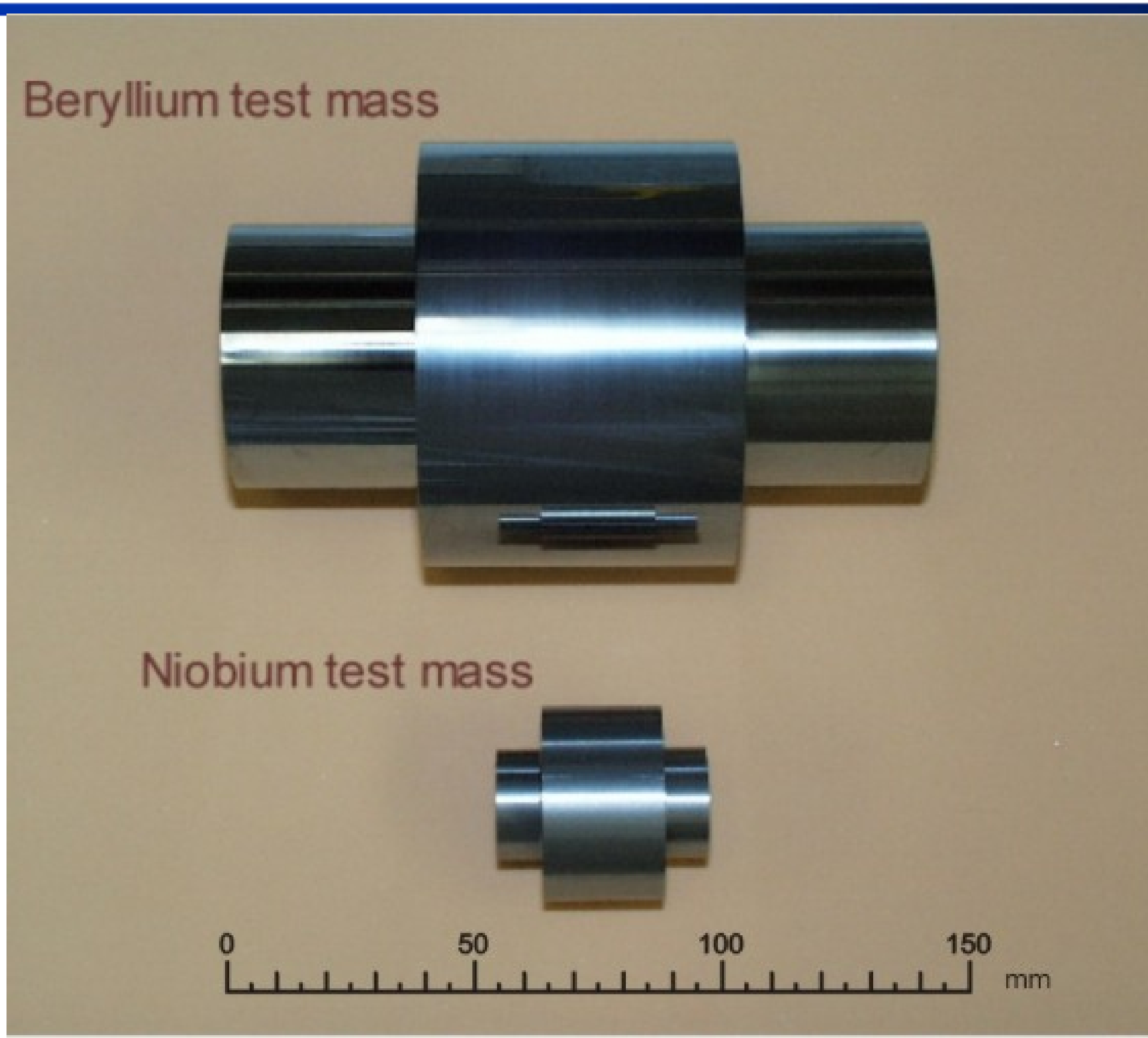

Background work, cont'd...

General Relativity and Gravitation **27**, 11 (1995) 1215–1229

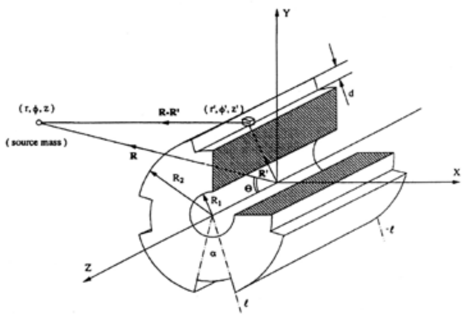

Figure 2. A cut-away view of grooves machined in a hollow cylindrical body.

$r' = a$, the outer radius $r' = b$, and $\phi = \pm\alpha/2$. The semi-length of the groove is l.

Firstly, integrating over z' (with no loss of generality taking $z > z'$)

$$V(\mathbf{R}) = -G\rho \sum_{m=0}^{\infty} \epsilon_m \int_{\phi'=-\alpha}^{\alpha} \cos[m(\phi-\phi')]d\phi' \int_{k=0}^{\infty} e^{-kz} J_m(kr) \times$$
$$\times \int_{r'=a}^{b} J_m(kr')r'\left[\frac{1}{k}e^{kz'}\right] dr'dk \qquad (3)$$

and using the well-known series expansions for the Bessel and exponential functions it is straightforward to show that

$$\int_a^b J_m(kr')r'dr' = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+m}(2n+m+2)n!(n+m)!} \times$$
$$\times (b^{2n+m+2} - a^{2n+m+2})k^{2n+m},$$

and

$$e^{kl} - e^{-kl} = \sum_{q=0}^{\infty} \frac{l^q}{q!}[1-(-1)^q]k^q = 2\sum_{q=0}^{\infty} \frac{l^{2q+1}k^{2q+1}}{(2q+1)!}$$

so that the potential $V(\mathbf{R})$ may be expanded as a triple summation over

Class. Quantum Grav. **13** (1996) 2041–2059

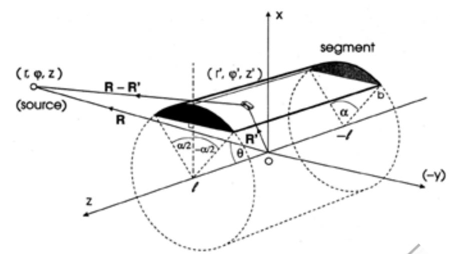

Figure 2. The coordinate system used for calculating the … interaction between a point source mass at position (r, φ, z), and the segment … shown shaded) of uniform mass density, ρ.

In general, we may make the following expan… of the Laplacian operator [2]:

$$\frac{1}{|\boldsymbol{R}-\boldsymbol{R}'|} = \sum_{m=0}^{\infty} \varepsilon_m \cos m \ldots$$

and, if the mass density of the f… y write the density of the solid segment to be $\rho(\boldsymbol{R}')$ = … he gravitational potential of the segment at an arbitrary … d by integration of equation (1) over the volume of the …

In the circular … in figure 2 the general point for the source mass is l… within the segment shown in the figure is labelled $\boldsymbol{R}'$ … volume of the segment is therefore to be carried ou… $= -\ell$ to $z' = \ell$, in azimuth from $\varphi' = -\alpha/2$ to $\varphi' = \alpha$ … $2)/\cos(\varphi')$ to $r' = b$. The gravitational potential of th… , can therefore be written as

$$\ldots \cos[m(\varphi-\varphi')]\,d\varphi' \int_{k=0}^{\infty} J_m(kr) \int_{r'=b\cos(\alpha/2)/\cos\varphi'}^{b} J_m(kr')r'$$
$$\ldots e^{-k|z-z'|}\,dz'\,dr'\,dk \qquad (2)$$

where G … universal gravitational constant ($= 6.67\times10^{-11}$ N m^2 kg^{-2}) and the Neumann factor $\varepsilon_m = 1$ ($m = 0$) and $\varepsilon_m = 2$ ($m > 0$).

Integrating firstly over z' (with no loss of generality taking $z > z'$)

$$V(\boldsymbol{R}) = -G\rho \sum_{m=0}^{\infty} \varepsilon_m \int_{\varphi'=-\alpha/2}^{\alpha/2} \cos[m(\varphi-\varphi')] \int_{k=0}^{\infty} e^{-kz} J_m(kr)$$
$$\times \int_{r'=b\cos(\alpha/2)/\cos\varphi'}^{b} J_m(kr')r'\left[\frac{1}{k}e^{kz'}\right]_{-\ell}^{\ell} dr'\,dk\,d\varphi' \qquad (3)$$

6 'Flats'

Conclusions

If there are *local* gravitational sources perturbing the GW test-masses, their

- axial effect will not be intuitively obvious; and the troublesome
  - Quadrupolar coupling is unlikely to be negligible out to distances > 0.4 m. However this form of Newtonian coupling
  - may be nulled through choice of test-mass dimensions ($\ell = \left(\sqrt{3}/2\right) b$); but
  - does this impact the test-mass coating noise adversely ?
  - Moment expansion is very useful (computationally fast).
- If test-mass dimensions must be retained, can any local, potentially interfering structures, be placed in and around the known Newtonian axial-coupling 'notches' of the test-masses ?
- Can each test-mass have 6 (rather than 2) equally-spaced longitudinal 'flats' ?