Calculating gravitational waveforms: examples DCC: LIGO-T1200476

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Contents

Abstract

We compute the GW waveforms for a few common astrophysical situations, using the equations and conventions of Kip Thorne's famous review article (Thorne, 1980).

1 Overview

In this article I compute the 'plus' and 'cross' polarisation components of gravitational waves for a few common astrophysical situations. I follow carefully the notation and equations of Thorne (1980). All of this is standard; the main idea is to spell out carefully the exact form of the equations, getting all the signs right, and making clear the conventions used. I repeat some material in the various examples, so that they can be read independently of oneanother; the reader need only look at section 2 and the section of direct interest to him/her.

In what follows, an equation number of the form (K4.3) means 'equation 4.3 of Thorne (1980)', while an equation number of the form (1, K4.3) means 'equation (1) of this note, or equivalently equation 4.3 of Thorne (1980)'.

2 Basic equations

I collect here the main equations for GW emission, some specific to the quadrpole ($l = 2$ case) for nearly-Newtonian sources.

Also, in the appendices, I give explicit forms for some of the key mathematical functions:

- Appendix A: The scalar spherical harmonics Y_{2m}, evaluated in terms of polar coordinates (θ, ϕ).
- Appendix B: The symmetric trace-free tensor basis $\mathcal{Y}^{2m}_{ab}$, evaluated with respect to a Cartesian basis $(\mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z})$.
- Appendix C: The tensor spherical harmonics $T^{\mathrm{E}2,2m}_{ab}$ and $T^{\mathrm{B}2,2m}_{ab}$, evaluated in terms of polar coordinates (θ, ϕ) and with respect to an orthonormal spherical polar basis $(\mathbf{e_r}, \mathbf{e_\theta}, \mathbf{e_\phi})$.
- Appendix D: The spin-weighted spherical harmonics $_{-2}Y^{2m}$, evaluated in terms of polar coordinates (θ, ϕ).

The fundamental quantities that appears in the wave generation sections of Thorne (1980) are the mass quadrupole moment scalars, related to the source's density field ρ by (K5.18b), or equivalently (K5.27a):

$$I_{2m} = \frac{16\pi\sqrt{3}}{15} \int \rho Y^*_{2m} r^2 \, dV, \tag{1}$$

and the current quadrupole of (K.18b), sourced by the momentum density $\rho\mathbf{v}$:

$$S_{2m} = \frac{32\pi\sqrt{2}}{15} \int \rho\mathbf{v} \cdot \mathbf{Y}^*_{B,2m} r^2 \, dV, \tag{2}$$

where Y_{lm} is the usual spherical harmonic, and $\mathbf{Y}_{B,lm}$ is a the magnetic-type 'pure spin vector harmonic', given by (K2.18):

$$\mathbf{Y}_{B,2m} = \frac{1}{6^{1/2}} \mathbf{r} \times \nabla \mathbf{Y}_{2m}. \tag{3}$$

The transverse traceless (TT) description of the GW field is given by (K4.3):

$$h^{\rm TT}_{ab}(t) = \frac{1}{r} \sum_m \ddot{I}^{2m} T^{\rm E2,2m}_{ab} + \ddot{S}^{2m} T^{\rm B2,2m}_{ab}. \tag{4}$$

The GW luminosity is given by (K4.16):

$$\frac{dE}{dt} = \frac{1}{32\pi} \sum_m < |^{(3)} I_{2m}|^2 > + < |^{(3)} S_{2m}|^2 > . \tag{5}$$

In most of what follows, we will evaluate $h^{\rm TT}_{ab}$ with respect to a spherical polar basis $(\mathbf{e_r}, \mathbf{e}_\theta, \mathbf{e}_\phi)$, so that the non-zero elements of $h^{\rm TT}_{ab}$ will lie in the $(\mathbf{e}_\theta, \mathbf{e}_\phi)$ plane, transverse to the direction of wave propagation. Then the polarisation components can be simply read-off from the matrix expressions for $h^{\rm TT}_{ab}$, with the understanding that the corresponding hypothetical detector, with respect to which the polarisations are defined, has its 1-arm along $\mathbf{e}_\theta$ and its 2-arm along $\mathbf{e}_\phi$. If one wants the polarisation components for a different detector, still lying in the transverse $r =$ constant space, but with arms along the orthonormal vectors $(\mathbf{e^1}, \mathbf{e^2})$, one simply contracts the metric perturbation with the tensors $e^+_{ab}/2$ and $e^\times_{ab}/2$:

$$\begin{aligned} h^+ &= \frac{1}{2} h^{\rm TT}_{ab} e^+_{ab}, & (6) \\ h^\times &= \frac{1}{2} h^{\rm TT}_{ab} e^\times_{ab}, & (7) \end{aligned}$$

where e^+_{ab} and $e^\times_{ab}$ are the polarisation tensors connected with the hypothetical detector:

$$\begin{aligned} e^+_{ab} &= (\mathbf{e^1 e^1} - \mathbf{e^2 e^2})_{ab}, & (8) \\ e^+_{ab} &= (\mathbf{e^1 e^2} + \mathbf{e^2 e^1})_{ab}. & (9) \end{aligned}$$

As an alternative to the mass quadrpole moment scalars I_{2m}, the gravitational wave emission can be related to the symmetric trace-free quadrupole moment tensor $\mathcal{I}_{ab}$. If we denote the mass quadrupolar tensor itself by I_{ab}, then

$$I_{ab} = \int \rho x_a x_b \, dV, \tag{10}$$

so that

$$\mathcal{I}_{ab} = [I_{ab}]^{\mathrm{STF}} = \int \rho (x_a x_b - \frac{1}{3}\delta_{ab} r^2) \, dV = I_{ab} - \frac{1}{3}\delta_{ab} I_{cc}, \tag{11}$$

where $r = (x_a x_a)^{1/2}$, the radius coordinate. Comparing with the moment of inertia tensor of rigid body mechanics:

$$I_{ab}^{\mathrm{MoI}} = \int \rho (\delta_{ab} r^2 - x_a x_b) \, dV = \delta_{ab} I_{cc} - I_{ab}. \tag{12}$$

One can go back-and-forth between the mass quadrupole moment scalars I_{2m} and the symmetric trace-free part of mass quadrupole tensor $\mathcal{I}_{ab}$ using (K4.6a) and (K4.7a):

$$\mathcal{I}_{ab} = \frac{1}{2\sqrt{3}} \sum_m I_{2m} \mathcal{Y}_{ab}^{2m}, \tag{13}$$

$$I_{2m} = \frac{16\pi\sqrt{3}}{15} (\mathcal{Y}_{ab}^{2m})^* \mathcal{I}_{ab}. \tag{14}$$

Writing out the second of these relations explicitly, and using a Cartesian coordinate system:

$$I_{2-2} = 2\sqrt{\frac{2\pi}{5}} [I_{xx} + 2i I_{xy} - I_{yy}], \tag{15}$$

$$I_{2-1} = 4\sqrt{\frac{2\pi}{5}} [I_{xz} + i I_{yz}], \tag{16}$$

$$I_{20} = -4\sqrt{\frac{\pi}{15}} [I_{xx} + I_{yy} - 2I_{zz}]. \tag{17}$$

Note that one can use either $\mathcal{I}_{ab}$ or I_{ab} here, as these formulae are invariant under addition of a pure trace to I_{ab}. These can then be used to construct the GW waveform and luminosity, as described above. This approach is particularly useful in rigid-body calculations, where the time dependent STF moment of inertia tensor I_{ab} can be obtained from its constant body-frame form by time-dependent active rotation. These relations might also be useful in a computer code, where the integrals over volume needed to compute I_{ab} are easily evaluated.

A slightly different formalism seems to be currently popular in the GW literature, making use of *spin-weighted spherical harmonics* ${}_sY^{lm}$ rather than tensor spherical harmonics. These are mentioned in Thorne (1980), with their relation to tensor spherical harmonics being given in (K2.38e). They are developed further in Kidder (2008), whose formalism we will follow here. We start with Kidder's equation (11):

$$h_+ - ih_\times = \sum_m h_{2m}\ {}_{-2}Y^{2m} \equiv \frac{1}{r}\sum_m H_{2m}\ {}_{-2}Y^{2m}, \tag{18}$$

where we follow Ott's DCC note in defining $H_{2m} = rh_{lm}$. The amplitudes H_{lm} play a similar role to the I_{lm} of Kip's multipole formalism; using Kidder's equation (19) we see that for mass quadrupoles the coefficients are related by

$$H_{2m} = \frac{1}{\sqrt{2}}\ddot{I}_{2m}. \tag{19}$$

These results can then be substituted into equation (18) to provide a slightly different route to that of equation (4) for calculating the polarisation components h_+ and $h_\times$.

The energy result can also be written in terms of the H_{2m}. Substituting equation (19) into equation (5) we have

$$\frac{dE}{dt} = \frac{1}{16\pi}\sum_m < |\dot{H}_{2m}|^2 >, \tag{20}$$

so the spin-weighted spherical harmonic-based luminosity calculation differs only trivially from that of equation (5), which was based on the scalar mass multipoles.

3 Example: A 'mountain' or bar-mode

Consider a body rotating rigidly at rate Ω with a quadrupolar deformation, emitting GWs at 2Ω. In the CW context, this is a 'mountain', in the burst context, it may be a 'bar mode'. Rather than starting with the quadrupole moment tensor I_{ab}, we will start with the density distribution $\rho(\mathbf{r}, t)$. This will provide a nice warm-up for the f-mode calculation of the next section. The relationship between this density-based description and the more usual description in terms of $I_{xx} - I_{yy}$, the asymmetry in the quadrupole moment tensor, will be derived in due course.

We will write the density as a spherical piece and a piece proportional to Y_{2-2}:

$$\rho(\mathbf{r}, t) = \rho_s(r) + \Re[\rho_{-2}(r) e^{i(\omega t + \Phi_{-2})} Y_{2-2}(\theta, \phi)], \tag{21}$$

where $\omega = 2\Omega$. Using $Y_{lm}(\theta, \phi) = Y_{lm}(\theta, 0) e^{im\phi}$ this becomes

$$\rho(\mathbf{r}, t) = \rho_s(r) + \Re[\rho_{-2}(r) e^{i(\omega t - 2\phi + \Phi_{-2})} Y_{2-2}(\theta, 0)] \tag{22}$$

or, extracting the real part,

$$\rho(\mathbf{r}, t) = \rho_s(r) + \rho_{-2}(r) \cos(\omega t - 2\phi + \Phi_{-2}) Y_{2-2}(\theta, 0). \tag{23}$$

A point of constant density rotates about Oz at a rate that can be found by looking at points of constant phase:

$$\frac{d}{dt}(\omega t - 2\phi + \Phi_{-2}) = 0 \rightarrow \dot{\phi}_{\text{pattern}} = \frac{\omega}{2} = \Omega, \tag{24}$$

confirming that the pattern speed is equal to the star's angular velocity. Having obtained a neat expression for the (real) density field, it is useful to rewrite in terms of complex exponentials:

$$\rho(\mathbf{r}, t) = \rho_s(r) + \frac{1}{2}\rho_{-2}(r)[e^{i(\omega t - 2\phi + \Phi_{-2})} + e^{-i(\omega t - 2\phi + \Phi_{-2})}] Y_{2-2}(\theta, 0). \tag{25}$$

Using the relation $Y_{22}(\theta, 0) = Y_{2-2}(\theta, 0)$ this becomes

$$\rho(\mathbf{r}, t) = \rho_s(r) + \frac{1}{2}\rho_{-2}(r)[e^{i(\omega t + \Phi_{-2})} Y_{2-2}(\theta, \phi) + e^{-i(\omega t + \Phi_{-2})} Y_{22}(\theta, \phi)]. \tag{26}$$

Having written the density field as the sum of two spherical harmonic functions, we can now make use of equation (1, K5.18a) to give the mass quadrupole moment scalars:

$$I_{2\pm 2} = \frac{8\pi\sqrt{3}}{15} e^{\mp i(\omega t + \Phi_{-2})} M_{-2}, \tag{27}$$

$$I_{2\pm 1} = I_{20} = 0, \tag{28}$$

where, for ease of notation, we have defined

$$M_{-2} \equiv \int \rho_{-2}(r) r^4 \, dr. \tag{29}$$

Note that, even though our density field consisted only of an $m = -2$ part, as defined by equation (21), the mass quadrupole moment scalars are non-zero for both $m = +2$ and $m = -2$. This is a consequence of the real part of Y_{2-2} (as per (21)) being the sum of a linear combination of Y_{22} and Y_{2-2} (as per equation (26)).

These mass quadrupole scalars can then be inserted into equation (4, K4.3): to give the GW field:

$$\begin{aligned} h_{ab}^{\mathrm{TT}} = & - \sqrt{\frac{2\pi}{15}} \omega^2 M_{-2} \Big\{ (1 + \cos^2\theta) \cos(\omega t - 2\phi + \Phi_{-2}) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ & + 2 \cos\theta \sin(\omega t - 2\phi + \Phi_{-2}) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Big\}, \end{aligned}$$

where the matrix elements span the $(\mathbf{e}_\theta, \mathbf{e}_\phi)$ space. The polarisation components with respect to the $(\mathbf{e}_\theta, \mathbf{e}_\phi)$ basis can then be read-off:

$$h_+ = -\frac{1}{r} \sqrt{\frac{2\pi}{15}} \omega^2 M_{-2} (1 + \cos^2\theta) \cos(\omega t - 2\phi + \Phi_{-2}), \tag{30}$$

$$h_\times = -\frac{1}{r} \sqrt{\frac{2\pi}{15}} \omega^2 M_{-2} 2 \cos\theta \sin(\omega t - 2\phi + \Phi_{-2}). \tag{31}$$

To make contact with the more familiar way of writing this in terms of the asymmetry in the mass quadrupole tensor I_{ab}, we can make use of equation (13, K4.6a) relating the STF form of the quadrupole moment tensor to the mass quadrupole scalars, to give:

$$\mathcal{I}_{ab} = \sqrt{\frac{2\pi}{15}} \omega^2 M_{-2} \begin{bmatrix} \cos(\omega t + \Phi_{-2}) & \sin(\omega t + \Phi_{-2}) & 0 \\ \sin(\omega t + \Phi_{-2}) & -\cos(\omega t + \Phi_{-2}) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{32}$$

As expected, the components of the (STF) quadrupole moment tensor, evaluated in the inertial frame, are time dependent. We can transform to the rotating frame. If a vector has components v_a in the inertial frame, then its components $\hat{v}_a$ in a frame that rotates at a rate Ω are given by:

$$\hat{v}_a = R_{ab} v_b, \tag{33}$$

where

$$R_{ab} = \begin{bmatrix} \cos\Omega t & \sin\Omega t & 0 \\ -\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{34}$$

For the SFT-quadrupole moment tensor

$$\hat{\mathcal{I}}_{ab} = R_{ac}R_{bd}\mathcal{I}_{cd} = (R\mathcal{I}R^{\mathrm{T}})_{ab}, \tag{35}$$

which leads to

$$\hat{\mathcal{I}}_{ab} = \sqrt{\frac{2\pi}{15}}\omega^2 M_{-2} \begin{bmatrix} \cos\Phi_{-2} & \sin\Phi_{-2} & 0 \\ \sin\Phi_{-2} & -\cos\Phi_{-2} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{36}$$

So, in the rotating frame, the components of $\mathcal{I}_{ab}$ are constant. If we specialise to the 'principal' frame, i.e. the rotating frame that is orientated onto the body, such that $\Phi_{-2} = 0$, we have

$$\hat{\mathcal{I}}^{\mathrm{p}}_{ab} = \sqrt{\frac{2\pi}{15}}\omega^2 M_{-2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{37}$$

from which we see

$$M_{-2} = \frac{1}{2}\sqrt{\frac{15}{2\pi}}(I^{\mathrm{p}}_{xx} - I^{\mathrm{p}}_{yy}). \tag{38}$$

where we have used $\mathcal{I}^{\mathrm{p}}_{xx} - \mathcal{I}^{\mathrm{p}}_{yy} = I^{\mathrm{p}}_{xx} - I^{\mathrm{p}}_{yy}$. Inserting this into the waveform of equations (30) and (31) and setting $\omega = 2\Omega$ leads to the more familiar expressions:

$$h_+ = -\frac{2}{r}\Omega^2(I^{\mathrm{p}}_{xx} - I^{\mathrm{p}}_{yy})(1+\cos^2\theta)\cos(2\Omega t - 2\phi + \Phi_{-2}), \tag{39}$$

$$h_\times = -\frac{2}{r}\Omega^2(I^{\mathrm{p}}_{xx} - I^{\mathrm{p}}_{yy})2\cos\theta\sin(2\Omega t - 2\phi + \Phi_{-2}). \tag{40}$$

This agrees with the results in the literature, e.g. equation (2) of Zimmermann & Szedenits (1979) (if one sets their wobble angle a to zero, and chooses $-2\phi + \Phi_{-2} = \pi$), or equations (47) and (48) of Jones (2010).

Now look at the GW luminosity, as given by equation (5, K4.16). Making use of the mass scalar quadrupole moments of equations (27) and (28) we obtain

$$\frac{dE}{dt} = \frac{4\pi}{75}|M_{-2}|^2\omega^6. \tag{41}$$

Inserting $\omega = 2\Omega$ and M_{-2} as given by equation (38) we obtain

$$\frac{dE}{dt} = \frac{32}{5}(I^{\rm p}_{xx} - I^{\rm p}_{yy})^2\Omega^6, \tag{42}$$

in agreement with equation (16.6.9) of Shapiro & Teukolsky (1983).

If we instead use the spin-weighted spherical harmonic formalism, we can use equation (19) to obtain the scalars H_{2m} from the I_{2m} of equations (27) and (28), giving

$$H_{2\pm2} = -\omega^2 \frac{8\pi\sqrt{3}}{15\sqrt{2}} e^{\mp i(\omega t + \Phi_{-2})} M_{-2}, \tag{43}$$

$$H_{2\pm1} = H_{20} = 0. \tag{44}$$

These can then be substituted into equation (18), together with the spin-weighted spherical harmonics of Appendix D, to give an expression for $(h_+ - ih_\times)$ consistent with the GW components of equations (30) and (31).

Finally, a more conventional approach would have taken as its starting point the quadrupole moment tensor in the rotating (body) frame, carried out a rotation to give its components I_{ab} in the inertial frame, and used (14, K4.7a)

$$I_{22} = 2\sqrt{\frac{2\pi}{5}}[I_{xx} - 2iI_{xy} - I_{yy}] = (I_{2-2})^*. \tag{45}$$

to find the mass quadrupole scalars. These can then be used as described above to give the wave field and luminosity. This of course leads to an identical result.

4 Example: f-modes

We want to start with a density perturbation, as would come out of a normal-mode calculation, and end up with the GW field in terms of its 'plus' and 'cross' polarisations.

Normal mode calculations generally take place on stellar backgrounds that are both stationary (i.e. not changing in time) and axisymmetric. The former means that the time-dependence can be written as $\sim e^{i\omega t}$, while the latter means that the ϕ-dependence can be written as $e^{im\phi}$. For the f-modes of spherical (i.e. non-rotating, non-magentic) stars, each mode is in fact associated with a single spherical harmonic $Y_{lm}(\theta, \phi) \sim e^{im\phi}$. The (real) density field is then decomposed into modes as:

$$\rho(\mathbf{r}, t) = \rho_{\rm s}(r) + \Re \sum_{lm} \rho_{lm}(r) e^{i(\omega_{lm} t + \Phi_{lm})} Y_{lm}(\theta, \phi), \tag{46}$$

where $\rho_s(r)$ is the density of the unperturbed star, ρ_{lm} the radial variation in the mode eigenfunction, ω_{lm} the (inertial frame) mode frequency, and Φ_{lm} are phase-constants. The form of ρ_{lm} and value of ω_{lm} would come out of a detailed mode calculation. The values of Φ_{lm} would depend upon how the glitch is initiated, i.e. the initial conditions. Note that ω_{lm} is also the gravitational wave frequency.

In fact, for perfectly spherical backgrounds, functions $\rho_{lm}(r)$ and the mode frequencies ω_{lm} (and gravitational wave damping times, not explicitly included here) are degenerate, i.e. independent of m, so the m-label need only be retained on the phase constants Φ_{lm}. However, for non-spherical (but still axisymmetric stars), e.g. rotating or magnetised ones, the situation is more complicated. Without going into details, each mode will have contributions from many different Y_{lm} terms, all of the same m, but different l-values. But those higher than quadrupole (i.e. $l > 2$) will generally be much weaker gravitational wave emitters, so we won't worry about them; we will include only $l = 2$ terms. The broken symmetry also breaks the degeneracy in the damping times, and more importantly for gravitational wave searches, the mode frequencies ω_{lm}. We will therefore retain the m-subscripts on the frequencies, to allow for this possibility, with the understanding that other (higher-than-quadrupole) contributions to the mode and the gravitational wave emission have been neglected.

We will now re-write equation (46), putting it in a more useful form for use in the multipole formalism. The spherical harmonics have the property that

$$Y_{lm}(\theta, \phi) = Y_{lm}(\theta, 0) e^{im\phi}, \tag{47}$$

so that

$$\rho(\mathbf{r}, t) = \rho_{\rm s}(r) + \Re \sum_{lm} \rho_{lm}(r) e^{i(\omega_{lm} t + m\phi + \Phi_{lm})} Y_{lm}(\theta, 0). \tag{48}$$

Extracting the real part we obtain

$$\rho(\mathbf{r}, t) = \rho_{\rm s}(r) + \sum_{lm} \rho_{lm}(r) \cos(\omega_{lm} t + m\phi + \Phi_{lm}) Y_{lm}(\theta, 0). \tag{49}$$

Finally, explicitly set $l = 2$, following the discussion above:

$$\rho(\mathbf{r}, t) = \rho_s(r) + \sum_m \rho_m(r) \cos(\omega_m t + m\phi + \Phi_m) Y_{2m}(\theta, 0). \qquad (50)$$

Note that each separate $m \neq 0$ mode can be thought of as a propagating density pattern. A point of constant density moves around with a *pattern speed* that can be found by following points of constant phase:

$$\frac{d}{dt}(\omega_{lm} t + m\phi + \Phi_{lm}) = 0 \rightarrow \dot{\phi}_{\text{pattern}} = -\frac{\omega_{lm}}{m}. \qquad (51)$$

Allowing both ω_{lm} and m to take either sign (positive or negative) would be redundant, as modes with parameters (ω, m) and $(-\omega, -m)$ have the same phase evolution. We will therefore adopt the convention that all mode frequencies are positive ($\omega_{lm} > 0$), so that the sign of m controls the sense of propagation of the density pattern; the pattern propagates in the $+\phi$ direction for $m < 0$, and in the $-\phi$ direction for $m > 0$. Modes with $m = 0$ are axisymmetric and so have no propagating density pattern.

Having obtained a real density field, we now want to insert this into Thorne's multipole formalism to compute the GW emission. This is best done by writing the trigonometric functions above in terms of complex exponentials:

$$\rho(\mathbf{r}, t) = \rho_s(r) + \frac{1}{2} \sum_m \rho_m(r) [e^{i(\omega_m t + m\phi + \Phi_m)} + e^{-i(\omega_m t + m\phi + \Phi_m)}] Y_{2m}(\theta, 0). \quad (52)$$

Using the relation

$$Y_{2m}(\theta, 0) = (-1)^m Y_{2-m}(\theta, 0) \qquad (53)$$

we find

$$\rho(\mathbf{r}, t) = \rho_s(r) + \frac{1}{2} \sum_m \rho_m(r) [e^{i(\omega_m t + \Phi_m)} Y_{2m}(\theta, \phi) + (-1)^m e^{-i(\omega_m t + \Phi_m)} Y_{2-m}(\theta, \phi)]. \qquad (54)$$

Having written ρ as the sum of spherical harmonic functions, it is now easy to calculate the scalar mass multipole moments using equation (1, K5.18a): to give

$$I_{2m}(t) = \frac{8\pi\sqrt{3}}{15} MR^2 [\alpha_m e^{i(\omega_m t + \Phi_m)} + (-1)^m \alpha_{-m} e^{-i(\omega_{-m} t + \Phi_{-m})}], \qquad (55)$$

where we have introduced a dimensionless mode amplitude α_m:

$$\alpha_m \equiv \frac{1}{MR^2} \int \rho_m(r) r^4 \, dr, \tag{56}$$

where M and R denote the stellar mass and radius. These amplitudes will be of order the fractional density perturbation at some point in the star, or equivalently the fractional change in radius, or equivalently the speed of fluid motion expressed in terms of the sound speed; the exact details depend upon the stellar background.

Note that this formalism inevitably mixes up the ρ_m contributions, i.e. the quantity I_{2m} that appears in Kip's multipole equations is made up from contributions form both the m and $-m$ parts of the density field. This is a consequence of the real part of Y_{lm}, which appears in equation (46) for ρ, being the sum of both Y_{lm} and Y_{l-m} terms, as given by equation (54).

The GW field can then be written in terms of equation (4, K4.3); the mass scalar multipole moments are given in equation (55), while I give the explicit form of the tensor spherical harmonics $T_{ab}^{\mathrm{E2},2m}$ in Appendix C, written out with respect to the spherical polar basis $(\mathbf{e_r}, \mathbf{e}_\theta, \mathbf{e}_\phi)$. A little algebra then leads to

$$
\begin{aligned}
rh^{\rm TT}_{ab} = & - \sqrt{\frac{2\pi}{15}}\alpha_2 MR^2\omega_2^2\Big\{(1+\cos^2\theta)\cos(\omega_2 t+2\phi+\Phi_2)\left[\begin{array}{cc}1 & 0\\ 0 & -1\end{array}\right] \\
& \qquad -2\cos\theta\sin(\omega_2 t+2\phi+\Phi_2)\left[\begin{array}{cc}0 & 1\\ 1 & 0\end{array}\right]\Big\}, \\
& - \sqrt{\frac{2\pi}{15}}\omega_{-2}^2 MR^2\alpha_{-2}\Big\{(1+\cos^2\theta)\cos(\omega_{-2} t-2\phi+\Phi_{-2})\left[\begin{array}{cc}1 & 0\\ 0 & -1\end{array}\right] \\
& \qquad +2\cos\theta\sin(\omega_{-2} t-2\phi+\Phi_{-2})\left[\begin{array}{cc}0 & 1\\ 1 & 0\end{array}\right]\Big\}, \\
& + \sqrt{\frac{2\pi}{15}}\omega_1^2 MR^2\alpha_1\Big\{-2\sin\theta\cos\theta\cos(\omega_1 t+\phi+\Phi_1)\left[\begin{array}{cc}1 & 0\\ 0 & -1\end{array}\right] \\
& \qquad +2\sin\theta\sin(\omega_1 t+\phi+\Phi_1)\left[\begin{array}{cc}0 & 1\\ 1 & 0\end{array}\right]\Big\}, \\
& + \sqrt{\frac{2\pi}{15}}\omega_{-1}^2 MR^2\alpha_{-1}\Big\{2\sin\theta\cos\theta\cos(\omega_{-1} t-\phi+\Phi_{-1})\left[\begin{array}{cc}1 & 0\\ 0 & -1\end{array}\right], \\
& \qquad +2\sin\theta\sin(\omega_{-1} t-\phi+\Phi_{-1})\left[\begin{array}{cc}0 & 1\\ 1 & 0\end{array}\right]\Big\} \\
& - 2\sqrt{\frac{\pi}{5}}\omega_0^2 MR^2\alpha_0\sin^2\theta\cos(\omega_0 t+\Phi_0)\left[\begin{array}{cc}1 & 0\\ 0 & -1\end{array}\right]. \qquad (57)
\end{aligned}
$$

where the matrices are written with respect to the $(\mathbf{e}_\theta, \mathbf{e}_\phi)$ basis. We can then easily read-off the polarisation components with respect to this basis:

$$h_+(m=+2) = -\frac{1}{r}\sqrt{\frac{2\pi}{15}}\omega_2^2 MR^2\alpha_2(1+\cos^2\theta)\cos(\omega_2 t+2\phi+\Phi_2), \quad (58)$$

$$h_\times(m=+2) = \frac{1}{r}\sqrt{\frac{2\pi}{15}}\omega_2^2 MR^2\alpha_2 2\cos\theta\sin(\omega_2 t+2\phi+\Phi_2), \quad (59)$$

$$h_+(m=+1) = -\frac{1}{r}\sqrt{\frac{2\pi}{15}}\omega_1^2 MR^2\alpha_1 2\sin\theta\cos\theta\cos(\omega_1 t+\phi+\Phi_1), \quad (60)$$

$$h_\times(m=+1) = \frac{1}{r}\sqrt{\frac{2\pi}{15}}\omega_1^2 MR^2\alpha_1 2\sin\theta\sin(\omega_1 t+\phi+\Phi_1), \quad (61)$$

$$h_+(m=0) = -2\sqrt{\frac{\pi}{5}}\omega_0^2 MR^2\alpha_0\sin^2\theta\cos(\omega_0 t+\Phi_0), \quad (62)$$

$$h_\times(m=0) = 0, \quad (63)$$

$$h_+(m=-1) = \frac{1}{r}\sqrt{\frac{2\pi}{15}}\omega_{-1}^2 MR^2\alpha_{-1} 2\sin\theta\cos\theta\cos(\omega_{-1} t-\phi+\Phi_{-1}), \quad (64)$$

$$h_\times(m=-1) = \frac{1}{r}\sqrt{\frac{2\pi}{15}}\omega_{-1}^2 MR^2\alpha_{-1} 2\sin\theta\sin(\omega_{-1} t-\phi+\Phi_{-1}), \quad (65)$$

$$h_+(m=-2) = -\frac{1}{r}\sqrt{\frac{2\pi}{15}}\omega_{-2}^2 MR^2\alpha_{-2}(1+\cos^2\theta)\cos(\omega_{-2} t-2\phi+\Phi_{-2}), \quad (66)$$

$$h_\times(m=-2) = -\frac{1}{r}\sqrt{\frac{2\pi}{15}}\omega_{-2}^2 MR^2\alpha_{-2} 2\cos\theta\sin(\omega_{-2} t-2\phi+\Phi_{-2}). \quad (67)$$

Clearly, the five different $l = 2$ modes show up neatly separated in the waveform, each with its own amplitude, frequency and phase constant $(\alpha_m, \omega_m, \Phi_m)$.

- **Is it worth seeing how the above results, in the appropriate limit, reduce to the waveform used in the Vela glitch paper?**

We can then calculate the energy emission using equation (5, K4.16); inserting the mass quadrupole moment scalars of equation (55) we find

$$\frac{dE}{dt} = \frac{4\pi}{75}\sum_m \omega_m^6 (MR^2\alpha_m)^2. \quad (68)$$

Note that cross-terms between modes of different m-values are eliminated by the time-averaging, even in the limit where their frequencies become the same.

If we had allowed for decay of the mode, we would have inserted a factor of e^{-t/τ_m} into the summation in the expressions for the density perturbation given above. Assuming that the decay timescale is long compared to the mode period, i.e. accepting errors related to the parameter $1/(\omega_m \tau_m) \ll 1$, the GW waveform of equation (57) simply acquires a factor of e^{-t/τ_m} in each separate m-term. The terms in the luminosity sumation each acquire a factor of e^{-2t/τ_m} to give

$$\frac{dE}{dt} = \frac{4\pi}{75} \sum_m \omega_m^6 (MR^2 \alpha_m)^2 e^{-2t/\tau_m}. \tag{69}$$

The total GW energy emitted is then

$$\Delta E \equiv \int_0^\infty \frac{dE}{dt}\, dt = \frac{2\pi}{75} \sum_m \omega_m^6 (MR^2 \alpha_m)^2 \tau_m. \tag{70}$$

This shows how the total radiated quadrupolar gravitational wave energy is shared over the five different m-modes. In the limit of a perfectly spherical background and no viscous dissipation, this must also be how the energy is shared over the modes themselves, i.e. how the initial excitation distributed its energy. This is useful, as one may have a total amount of, say, 'glitch energy', that one wants to distribute over the modes; the above sum shows how it can be divided out. (As a consistency check, note that, for gravitational wave damping of f-modes, $\tau_m \sim 1/(MR^2 \omega_m^4)$, so that each term in the energy sum is of the form $\sim MR^2 \alpha_m^2$, the expected scaling.) For such a perfectly spherical star, the τ_m and ω_m quantities are degenerate (i.e. independent of m), so this amounts to sharing out the energy according to the sum of the α_m^2 factors.

However, for non-spherical backgrounds, where this degeneracy is broken, the modes will have pieces with $l > 2$, and our calculation misses the gravitational wave energy corresponding to this $l > 2$ emission, so the terms in the sum of equation (70) will be somewhat smaller than the original energy deposited in each m-mode. One can still use the above sum to distribute energy for such non-spherical stars, but with the understanding that it is the radiated quadrupolar gravitational wave energy that is being fixed, not the actual energy initially deposited in the mode. With the tools described here, this is, I believe, the best that can be done.

As a (partial) check on the waveform we can use the spin-weighted spherical harmonic formalism. Using equation (19) that relates the coefficients

I_{2m} and H_{2m} we have

$$H_{2m} = -\frac{8\pi\sqrt{3}}{15\sqrt{2}}[\omega_m^2 MR^2\alpha_m e^{i(\omega_m t+\Phi_m)} + (-1)^m \omega_{-m}^2 MR^2 \alpha_{-m} e^{-i(\omega_{-m}t+\Phi_{-m})}]. \tag{71}$$

Substituting into equation (18) for $h_+ - ih_\times$, and making use of the spin-weighted spherical harmonics$_{-2}Y^{2m}$ as given in Appendix D, we can verify equation (57).

We can relate our results to the SFT quadrupole moment tensor using equation (13, K4.60). Some algebra leads to the following result:

$$\begin{aligned}
\mathcal{I}_{ab} = & \sqrt{\frac{2\pi}{15}}MR^2\alpha_2 \begin{bmatrix} \cos(\omega_2 t+\Phi_2) & -\sin(\omega_2 t+\Phi_2) & 0 \\ -\sin(\omega_2 t+\Phi_2) & -\cos(\omega_2 t+\Phi_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
+ & \sqrt{\frac{2\pi}{15}}MR^2\alpha_{-2} \begin{bmatrix} \cos(\omega_{-2} t+\Phi_{-2}) & \sin(\omega_{-2} t+\Phi_{-2}) & 0 \\ \sin(\omega_{-2} t+\Phi_{-2}) & -\cos(\omega_{-2} t+\Phi_{-2}) & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
- & \sqrt{\frac{2\pi}{15}}MR^2\alpha_1 \begin{bmatrix} 0 & 0 & \cos(\omega_1 t+\Phi_1) \\ 0 & 0 & -\sin(\omega_1 t+\Phi_1) \\ \cos(\omega_1 t+\Phi_1) & -\sin(\omega_1 t+\Phi_1) & 0 \end{bmatrix} \\
+ & \sqrt{\frac{2\pi}{15}}MR^2\alpha_{-1} \begin{bmatrix} 0 & 0 & \cos(\omega_{-1} t+\Phi_{-1}) \\ 0 & 0 & \sin(\omega_{-1} t+\Phi_{-1}) \\ \cos(\omega_{-1} t+\Phi_{-1}) & \sin(\omega_{-1} t+\Phi_{-1}) & 0 \end{bmatrix} \\
+ & \frac{2}{3}\sqrt{\frac{\pi}{5}}MR^2\alpha_0\cos(\omega_0 t+\Phi_0) \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}
\end{aligned} \tag{72}$$

Each individual $m \neq 0$ piece of the tensor can be made time-independent by going into the frame which rotates at the appropriate pattern speed. If we call the individual pieces $\mathcal{I}_{ab}^m$ we find:

In the frame that rotates at rate $\dot{\phi}_\mathrm{p} = -\omega_2/2$:

$$\mathcal{I}_{ab}^{m=2} = \sqrt{\frac{2\pi}{15}}MR^2\alpha_2 \begin{bmatrix} \cos\Phi_2 & -\sin\Phi_2 & 0 \\ -\sin\Phi_2 & -\cos\Phi_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{73}$$

In the frame that rotates at rate $\dot{\phi}_{\rm p} = \omega_{-2}/2$:

$$\mathcal{I}_{ab}^{m=-2} = \sqrt{\frac{2\pi}{15}} M R^2 \alpha_{-2} \begin{bmatrix} \cos\Phi_{-2} & \sin\Phi_{-2} & 0 \\ -\sin\Phi_{-2} & -\cos\Phi_{-2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{74}$$

In the frame that rotates at rate $\dot{\phi}_{\rm p} = -\omega_1$:

$$\mathcal{I}_{ab}^{m=1} = -\sqrt{\frac{2\pi}{15}} M R^2 \alpha_1 \begin{bmatrix} 0 & 0 & \cos\Phi_1 \\ 0 & 0 & -\sin\Phi_1 \\ \cos\Phi_1 & -\sin\Phi_1 & 0 \end{bmatrix} \tag{75}$$

In the frame that rotates at rate $\dot{\phi}_{\rm p} = \omega_{-1}$:

$$\mathcal{I}_{ab}^{m=-1} = \sqrt{\frac{2\pi}{15}} M R^2 \alpha_{-1} \begin{bmatrix} 0 & 0 & \cos\Phi_{-1} \\ 0 & 0 & \sin\Phi_{-1} \\ \cos\Phi_{-1} & \sin\Phi_{-1} & 0 \end{bmatrix} \tag{76}$$

In the inertial (or any other) frame:

$$\mathcal{I}_{ab}^{m=0} = \frac{2}{3}\sqrt{\frac{\pi}{5}} M R^2 \alpha_0 \cos(\omega_0 t + \Phi_0) \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{77}$$

I am not sure how useful or insightful these last results are!

5 Example: triaxial star, spinning steadily about an axis different from a principal axis

As discussed in Jones (2010), superfluid pinning can allow a triaxial star to spin *steadily* about an axis, with the axis fixed in space and fixed with respect to the star, even when that axis does not coincide with a principal axis of the star's moment of inertia tensor. The waveform was presented in Jones (2010). Here is a detailed derivation of this waveform, making use of the multipole moment formalism.

Given that the star rotates rigidly, the body-frame quadrupole moment of inertia tensor can be written as

$$I_{ab}^{\rm BF} = \begin{bmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{bmatrix} \tag{78}$$

The superfluid pinning allows the star to rotate about an axis whose orientation relative to the inertial frame is specified by the Euler angles:

$$\theta = \text{constant} \quad (79)$$

$$\phi = \Omega t + \phi_0 \quad (80)$$

$$\psi = \text{constant} \quad (81)$$

where ϕ_0 is the position of the line of nodes at $t = 0$. (The 'line of nodes' is the intersection of the inertial and body frame xy planes; see Figure 1 of Jones (2010)). We need to find I_{ab} with respect to the inertial frame, and then plug these results into equations (15)–(17) to find the mass quadrupole scalars I_{2m}.

Let R_{ab} denote the matrix that carries out an *active* rotation from the inertial xyz axes to the body-frame axes. Then, to produce a star whose orientation is specified by the three standard Euler angles (θ, ϕ, ψ), we need to first perform a rotation of ψ about the inertial z-axis, then a rotation of θ about the inertial x-axis, and finally a rotation of ϕ about the inertial z-axis. In an obvious notation, we then have

$$R_{ab} = [R^z(\phi) R^x(\theta) R^z(\psi)]_{ab} \quad (82)$$

so that

$$I_{ab}(t) = (R I^{\rm BF} R^T)_{ab} \quad (83)$$

where R^T is the transpose of R. Strictly, is the is STF quadrupole moment tensor that should appear in our formulae. However, we can reduce the amount of algebra we need perform by writing

$$\mathcal{I}^{\rm BF}_{ab} = \begin{bmatrix} 0 & & \\ & I_{21} & \\ & & I_{31} \end{bmatrix} + \frac{1}{3}(2I_1 - I_2 - I_3) \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad (84)$$

and noting the second term is a pure trace, and so will not affect the scalar quadrupole moments. We therefore need only transform the first term:

$$\Delta\mathcal{I}^{\rm BF}_{ab} = \begin{bmatrix} 0 & & \\ & I_{21} & \\ & & I_{31} \end{bmatrix} \quad (85)$$

i.e. we need to evaluate

$$\Delta I_{ab} = (R \Delta I^{BF} R^T)_{ab} \quad (86)$$

$$= \left\{ R^z(\phi) R^x(\theta) R^z(\psi) \Delta I^{\rm BF} [R^z(\phi)]^{\rm T} [R^z(\theta)]^{\rm T} [R^z(\phi)]^{\rm T} \right\}_{ab} \quad (87)$$

The relevant active rotation matrices are:

$$R^z_{ab}(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (88)$$

$$R^x_{ab}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad (89)$$

$$R^z_{ab}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (90)$$

Crunching through the algebra leads to

$$\begin{aligned}
\Delta I_{xx} &= 2I_{21}\sin\psi\cos\psi\cos\theta\sin\phi\cos\phi \\
&\quad + I_{21}(\sin^2\psi\cos^2\phi + \cos^2\psi\cos^2\theta\sin^2\phi) + I_{31}\sin^2\theta\sin^2\phi \quad (91) \\
\Delta I_{xy} &= I_{21}\sin\psi\cos\psi\cos\theta(\sin^2\phi - \cos^2\phi) \\
&\quad + I_{21}(\sin^2\psi - \cos^2\psi\cos^2\theta)\sin\phi\cos\phi - I_{31}\sin^2\theta\sin\phi\cos\phi \quad (92) \\
\Delta I_{xz} &= -I_{21}\sin\psi\cos\psi\sin\theta\cos\phi \\
&\quad - (I_{21}\cos^2\psi - I_{31})\sin\theta\cos\theta\sin\phi \quad (93) \\
\Delta I_{yy} &= -2I_{21}\sin\psi\cos\psi\cos\theta\sin\phi\cos\phi \\
&\quad + I_{21}(\sin^2\psi\sin^2\phi + \cos^2\psi\cos^2\theta\cos^2\phi) + I_{31}\sin^2\theta\cos^2\phi \quad (94) \\
\Delta I_{yz} &= -I_{21}\sin\psi\cos\psi\sin\theta\sin\phi \\
&\quad + (I_{21}\cos^2\psi - I_{31})\sin\theta\cos\theta\cos\phi \quad (95) \\
\Delta I_{zz} &= I_{21}\cos^2\psi\sin^2\theta + I_{31}\cos^2\theta
\end{aligned}$$

The mass quadrupole scalars are then:

$$\begin{aligned}
I_{2-2} &= 2\sqrt{\frac{2\pi}{5}}e^{2i\phi}[I_{21}(\sin^2\psi - \cos^2\psi\cos^2\theta) - I_{31}\sin^2\theta - 2iI_{21}\sin\psi\cos\psi\cos\theta] \\
I_{2-1} &= 2\sqrt{\frac{2\pi}{5}}e^{i\phi}[-I_{21}\sin 2\psi\sin\theta + i(I_{21}\cos^2\psi - I_{31})\sin 2\theta] \\
I_{20} &= -4\sqrt{\frac{\pi}{15}}\big\{I_{21}[\sin^2\psi + \cos^2\psi(\cos^2\theta - 2\sin^2\theta)] + I_{31}(\sin^2\theta - 2\cos^2\theta)\big\}
\end{aligned}$$

The gravitational wave field is then given by equation (4, K4.3), in terms of the above quadrupole moment scalars and the tensor spherical harmonics of Appendix C. In the latter we use the notation $T_{ab}^{\mathrm{E2},2m} = T_{ab}^{\mathrm{E2},2m}(\iota, \phi_{\mathrm{obs}})$, i.e. we place out observer at a location $(\iota, \phi_{\mathrm{obs}})$, as measured using polar coordinates defined with respect to the inertial frame. We then find:

$$\begin{aligned}
\frac{rh_{ab}^{\mathrm{TT}}}{\Omega^2} = \quad & e_{ab}^{+}(1+\cos^2\iota)2\Big\{-[I_{21}(\sin^2\psi - \cos^2\psi\cos^2\theta) - I_{31}\sin^2\theta)]\cos 2\phi_{\mathrm{gw}} \\
& -I_{21}\sin 2\psi\cos\theta\sin 2\phi_{\mathrm{gw}}\Big\} \\
+ \quad & e_{ab}^{\times}4\cos\iota\Big\{-[I_{21}(\sin^2\psi - \cos^2\psi\cos^2\theta) - I_{31}\sin^2\theta]\sin 2\phi_{\mathrm{gw}} \\
& +I_{21}\sin 2\psi\cos\theta\cos 2\phi_{\mathrm{gw}}\Big\} \\
+ \quad & e_{ab}^{+}\sin\iota\cos\iota\Big\{-I_{21}\sin 2\psi\sin\theta\cos\phi_{\mathrm{gw}} \\
& -(I_{21}\cos^2\psi - I_{31})\sin 2\theta\sin\phi_{\mathrm{gw}}\Big\} \\
+ \quad & e_{ab}^{\times}\sin\iota\Big\{-I_{21}\sin 2\psi\sin\theta\sin\phi_{\mathrm{gw}} \\
& +(I_{21}\cos^2\psi - I_{31})\sin 2\theta\cos\phi_{\mathrm{gw}}\Big\} \qquad (96)
\end{aligned}$$

where the gravitational phase ϕ_{gw} is given by

$$\phi_{\mathrm{gw}} = \Omega t + \phi_0 - \phi_{\mathrm{obs}} \qquad (97)$$

and for ease of notation we have defined the two STF basis tensors (i.e. 'polarisation tensors'), built out of the spherical polar basis vectors $(\mathbf{e}_\theta, \mathbf{e}_\phi)$:

$$e_{ab}^{+,\,\mathrm{polar}} = \mathbf{e}_{\theta a}\mathbf{e}_{\theta b} - \mathbf{e}_{\phi_a}\mathbf{e}_{\phi_b} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad (98)$$

$$e_{ab}^{\times,\,\mathrm{polar}} = \mathbf{e}_{\theta a}\mathbf{e}_{\phi_b} + \mathbf{e}_{\phi_a}\mathbf{e}_{\theta b} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad (99)$$

Equation (96) above give the transverse-traceless metric perturbation for an observer at an arbitrary location $(\iota, \phi_{\mathrm{obs}})$ a distance r from the source. The polarisation components with respect to $(\mathbf{e}_\theta, \mathbf{e}_\phi)$ can then be easily read-off.

The above results can be checked against those of Jones (2010), which were arrived by by slightly different means. First compare the mass quadrupole scalars. The normalisation of the I_{2m} used here and defined in (1) differs from the Q_{2m} of equation (38) of Jones (2010), which were defined to be

$$Q_{2m} = \int_V \rho(\mathbf{r}, t)(Y^{2m})^*\, dV \qquad (100)$$

so that we would expect

$$I_{2m} = \frac{16\pi\sqrt{3}}{15} Q_{2m} \qquad (101)$$

Allowing for this, and also for the fact that the quantities I_{21} and I_{31} in Jones (2010) refer to differences in the moment of inertia tensor, not the quadrupole moment tensor, and so are of opposite sign (see equation (12)) to the I_{21} and I_{31} that appear here, we find that the expressions for Q_{2m} of Jones (2010) are in fact *too large* by a factor of 2. In other words, an additional factor of 1/2 needs to inserted into (40) and (41) of Jones (2010). An additional factor of 1/2 also needs to be inserted in the expressions for $|Q_{2m}|$ in the final equalities of equations (56)–(58) of Jones (2010). This appears to be a typographical error that propagates no further; the corresponding equations in my hand written notes were in fact correct.

We can also check the formulae for $h^+, h^\times$ given in Jones (2010). To do so, we need to put $\phi_{\rm obs} = -\pi/2$, $\phi_0 = 0$, and again remember that the quantities I_{21} and I_{31} in Jones (2010) are of opposite sign to those used here. Also, the $h^+, h^\times$ of Jones (2010) refer to a detector with its 1-arm along $\mathbf{e^1} = \mathbf{e}^\phi$, and its 2-arm along $\mathbf{e^2} = -\mathbf{e}^\theta$. This means we need to use the projections of equations (6) and (7) to extract the polarisation components of Jones (2010):

$$h^{+,{\rm J10}} = \frac{1}{2} h^{\rm TT}_{ab} (\mathbf{e^1}\mathbf{e^1} - \mathbf{e^2}\mathbf{e^2})_{ab} = \frac{1}{2} h^{\rm TT}_{ab} (\mathbf{e}^\phi\mathbf{e}^\phi - \mathbf{e}^\theta\mathbf{e}^\theta)_{ab} \tag{102}$$

$$h^{\times,{\rm J10}} = \frac{1}{2} h^{\rm TT}_{ab} (\mathbf{e^1}\mathbf{e^2} + \mathbf{e^2}\mathbf{e^1})_{ab} = \frac{1}{2} h^{\rm TT}_{ab} (-\mathbf{e}^\phi\mathbf{e}^\theta - \mathbf{e}^\theta\mathbf{e}^\phi)_{ab} \tag{103}$$

Carrying out all these operations does indeed lead to the polarisation components of equations (42)–(44) of Jones (2010), which I won't bother writing out again here.

6 Example: r-modes

In most of the literature, the r-mode velocity field is written as

$$\delta \mathbf{v} = \alpha \Omega \frac{r^2}{R} \mathbf{Y}_{B,22} e^{i\omega t}, \tag{104}$$

with ω the mode frequency as viewed from the inertial frame

$$\omega = -\frac{4}{3}\Omega, \tag{105}$$

where Ω is the angular velocity of the star's rotation. The perturbation is then proportional to $e^{i(2\phi + \omega t)}$ and so has a positive pattern speed relative to the inertial frame of

$$\dot{\phi}_{\text{pattern, I}} = -\frac{\omega}{2} = +\frac{2}{3}\Omega, \tag{106}$$

giving a retrograde pattern motion as viewed from the rotting frame:

$$\dot{\phi}_{\text{pattern, R}} = \dot{\phi}_{\text{pattern, I}} - \Omega = -\frac{1}{3}\Omega. \tag{107}$$

The velocity field given above is complex. To make contact with reality, we need to specify how this makes contact with a purely real velocity field. We will write

$$\delta \mathbf{v}_R = \Re[\alpha_R \Omega \frac{r^2}{R} \mathbf{Y}_{B,22} e^{i\omega t}], \tag{108}$$

where the R subscript notation is to remind us that this is a definition of the velocity of the perturbation that is purely real. I am calling the amplitude α_R, not α, to allow for a slight difference between the α used in the literature and the α_R introduced here.

We then have

$$\delta \mathbf{v}_R = \frac{1}{2}\alpha_R \Omega \frac{r^2}{R}[\mathbf{Y}_{B,22} e^{i\omega t} + \mathbf{Y}^*_{B,22} e^{-i\omega t}]. \tag{109}$$

Using the property $\mathbf{Y}^*_{B,22} = \mathbf{Y}_{B,2,-2}$ this becomes

$$\delta \mathbf{v}_R = \frac{1}{2}\alpha_R \Omega \frac{r^2}{R}[\mathbf{Y}_{B,22} e^{i\omega t} + \mathbf{Y}_{B,2,-2} e^{-i\omega t}], \tag{110}$$

showing that the (real) velocity field is a linear combination of the $m = \pm 2$ magnetic vector harmonics, weighted by factors $e^{\pm i\omega t}$. [An an aside, one can easily show that this leads to the explicit result

$$\delta \mathbf{v}_R = \frac{1}{4}\sqrt{\frac{15}{\pi}}\alpha_R \Omega \frac{r^2}{R} \sin\theta [\mathbf{e}_\theta \sin(\omega t + 2\phi) + \cos\theta \mathbf{e}_\phi \cos(\omega t + 2\phi)], \tag{111}$$

although this isn't terribly illuminating.]

We can project out the two non-zero current quadrupole scalars using equation (2) to give

$$S_{2,\pm 2} = \frac{16\sqrt{2}}{15}\pi \alpha_R M R^3 \Omega \tilde{J} e^{\pm i\omega t}, \tag{112}$$

where $\tilde{J}$ is the integral introduced in Owen et al. (1998):

$$\tilde{J} = \frac{1}{MR^4} \int \rho r^6 \, dr. \tag{113}$$

We can then write down the gravitational wave field in TT gauge using equation (4) to give the purely real result

$$h_{ab}^{\mathrm{TT}} = \frac{1}{r}\sqrt{\frac{\pi}{5}} MR^3 \tilde{J} \alpha_R \omega^3 \left\{ 2\cos\theta \sin(\omega t + 2\phi) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right.$$
$$\left. -(1+\cos^2\theta)\cos(\omega t + 2\phi) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \tag{114}$$

Setting $\theta = 0$, we can read-off the h_0 favoured when discussing CW sources:

$$h_0 = \frac{1}{r} 2 \sqrt{\frac{\pi}{5}} MR^3 \tilde{J} \alpha_R \omega^3. \tag{115}$$

We can calculate the gravitational wave luminosity using equation (5):

$$\dot{E}_{\mathrm{GW}} = \frac{2}{25} \pi \alpha_R^2 M^2 R^6 \omega^8 \tilde{J}^2. \tag{116}$$

Note that the two equations above can be combined to give the result

$$\dot{E}_{\mathrm{GW}} = \frac{1}{10} \omega^2 r^2 h_0^2, \tag{117}$$

which is, I believe, a rather general result, true for any sort of quadrupolar radiation (whether from mass or current quadrupoles), for a gravitational wave field of frequency ω, with h_0 the amplitude of the circular polarisation along the z-axis.

6.1 Comparison with other formulae in the literature

In Owen (2010), Ben writes the r-mode velocity perturbation in the usual complex notation of equation (104), i.e. proportional to $\mathbf{Y}_{B,22}$ only, with no $\mathbf{Y}_{B,2,-22}$ piece. He then uses equation (2) to compute S_{22} in terms of α, with $S_{2,-2}$ zero. He then inserts his S_{22} into equation (5) to give the gravitational wave luminosity as a quadratic function of α. This can then be combined with equation (117) to give h_0 as a function of α. The main results are then:

$$h_0 = \frac{1}{r} 2 \sqrt{\frac{2\pi}{5}} M R^3 \tilde{J} \alpha \omega^3, \tag{118}$$

$$\dot{E}_{\rm GW} = \frac{4}{25} \pi \alpha^2 M^2 R^6 \omega^8 \tilde{J}^2. \tag{119}$$

These should be contrasted with equations (115) and (116) above, which we reprodce here:

$$h_0 = \frac{1}{r} 2 \sqrt{\frac{\pi}{5}} M R^3 \tilde{J} \alpha_R \omega^3, \tag{120}$$

$$\dot{E}_{\rm GW} = \frac{2}{25} \pi \alpha_R^2 M^2 R^6 \omega^8 \tilde{J}^2. \tag{121}$$

The quantities agree if one sets

$$\alpha_R = \sqrt{2} \alpha. \tag{122}$$

As an aside, note that our expressions for the 'intermediate' quantities S_{2m} are quite different. I have

$$S^{\rm Ian}_{2,\pm 2} = \frac{16\sqrt{2}}{15} \pi \alpha_R M R^3 \Omega \tilde{J} e^{\pm i\omega t}, \tag{123}$$

while Ben has

$$S^{\rm Ben}_{2,2} = -\frac{32\sqrt{2}}{15} \pi \alpha M R^3 \Omega \tilde{J} e^{i\omega t}, \tag{124}$$

$$S^{\rm Ben}_{2,-2} = 0. \tag{125}$$

When one substitutes $\alpha_R = \sqrt{2}\alpha$, one sees

$$S^{\rm Ben}_{22} = -\sqrt{2}\, S^{\rm Ian}_{2,2}, \tag{126}$$

$$S^{\rm Ben}_{2,-2} = 0. \tag{127}$$

The upshot of all this is that if one choose the convention of equations (118) and (119) above, then the corresponding real velocity perturbation is given by

$$\delta \mathbf{v}_R = \sqrt{2}\, \Re[\alpha \Omega \frac{r^2}{R} \mathbf{Y}_{B,22} e^{i\omega t}]. \tag{128}$$

Note the factor of $\sqrt{2}$ on the right hand side.

A Explicit form for the spherical harmonics Y_{2m}

Following the usual convention:

$$
\begin{aligned}
Y_{2-2} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{-2i\phi}, &(129)\\
Y_{2-1} &= \frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{-i\phi}, &(130)\\
Y_{20} &= \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1), &(131)\\
Y_{l-m} &= (-1)^m Y^*_{lm}. &(132)
\end{aligned}
$$

B Explicit form for the symmetric trace-free tensors $\mathcal{Y}^{2m}_{ab}$

The symmetric trace-free tensors $\mathcal{Y}^{lm}_{a_1,a_2,\dots a_l}$ are defined by (K2.12). They have the property of generating the spherical harmonics, as in (K2.11):

$$Y_{lm} = \mathcal{Y}^{lm}_{a_1,a_2,\dots a_l} n_{a_1} n_{a_2} \dots n_{a_l}, \tag{133}$$

where n_a is the radial unit vector. For $l = 2$ they are second rank tensors. Evaluating with respect to a Cartesian basis leads to:

$$
\begin{aligned}
\mathcal{Y}^{2-2}_{ab} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\begin{bmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, &(134)\\
\mathcal{Y}^{2-1}_{ab} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{bmatrix}, &(135)\\
\mathcal{Y}^{20}_{ab} &= \frac{1}{4}\sqrt{\frac{5}{\pi}}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, &(136)\\
\mathcal{Y}^{l-m}_{ab} &= (-1)^m(\mathcal{Y}^{lm}_{ab})^*. &(137)
\end{aligned}
$$

C Explicit form for tensor spherical harmonics $T_{ab}^{\mathrm{E2},2m}$

We give here explicit forms for the $T_{ab}^{\mathrm{E2},2m}$ and $T_{ab}^{\mathrm{B2},2m}$ tensor spherical harmonics. The former are needed to write down the GW field from a mass quadrupole, and the latter for the field from a mass-current quadrupole. We work in a spherical polar basis $(\mathbf{e_r}, \mathbf{e}_\theta, \mathbf{e}_\phi)$, so that $\mathbf{e_r}$ is the longitudinal direction, and $(\mathbf{e}_\theta, \mathbf{e}_\phi)$ span the transverse space.

Using (K2.30d), or alternatively (K2.39e):

$$T_{ab}^{\mathrm{E2},2-2} = \frac{1}{8}\sqrt{\frac{5}{2\pi}} e^{-2i\phi} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1+\cos^2\theta & -2i\cos\theta \\ 0 & -2i\cos\theta & -(1+\cos^2\theta) \end{bmatrix}, \tag{138}$$

$$T_{ab}^{\mathrm{E2},2-1} = \frac{1}{4}\sqrt{\frac{5}{2\pi}} e^{-i\phi} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta\cos\theta & i\sin\theta \\ 0 & i\sin\theta & \sin\theta\cos\theta \end{bmatrix}, \tag{139}$$

$$T_{ab}^{\mathrm{E2},20} = \frac{1}{8}\sqrt{\frac{15}{\pi}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sin^2\theta & 0 \\ 0 & 0 & -\sin^2\theta \end{bmatrix}. \tag{140}$$

Using (K2.30f), or alternatively (K2.39f):

$$T_{ab}^{\mathrm{B2},2-2} = \frac{1}{8}\sqrt{\frac{5}{2\pi}} e^{-2i\phi} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2i\cos\theta & 1+\cos^2\theta \\ 0 & 1+\cos^2\theta & -2i\cos\theta \end{bmatrix}, \tag{141}$$

$$T_{ab}^{\mathrm{B2},2-1} = \frac{1}{4}\sqrt{\frac{5}{2\pi}} e^{-i\phi} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -i\sin\theta & -\sin\theta\cos\theta \\ 0 & -\sin\theta\cos\theta & i\sin\theta \end{bmatrix}, \tag{142}$$

$$T_{ab}^{\mathrm{B2},20} = \frac{1}{8}\sqrt{\frac{15}{\pi}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \sin^2\theta \\ 0 & \sin^2\theta & 0 \end{bmatrix}. \tag{143}$$

The $m > 0$ tensors can be obtained from the above using (K2.36b)

$$T_{ab}^{\mathrm{J2},lm} = (-1)^m (T_{ab}^{\mathrm{J2},l-m})^*, \tag{144}$$

where $J = (E, B)$.

D Explicit form for spin-weighted spherical harmonics ${}_{-2}Y^{2m}$

We give here explicit for for the $s = -2$ spin-weighted $l = 2$ spherical harmonics. They can be computed using equations (4) and (5) of Kidder (2008). They can also be evaluated using equation (3.1) of Goldberg et al. (1967), but only if one inserts an extra factor of $(-1)^m$ into their formula, to ensure the usual normalisation convention for spin-weight zero spherical harmonics is used.

$$
{}_{-2}Y^{2-2} = \frac{1}{8}\sqrt{\frac{5}{\pi}}(1-\cos\theta)^2 e^{-2i\phi} \quad (145)
$$

$$
{}_{-2}Y^{2-1} = \frac{1}{4}\sqrt{\frac{5}{\pi}}\sin\theta(1-\cos\theta)e^{-i\phi} \quad (146)
$$

$$
{}_{-2}Y^{20} = \frac{1}{8}\sqrt{\frac{30}{\pi}}\sin^2\theta \quad (147)
$$

$$
{}_{-2}Y^{21} = \frac{1}{4}\sqrt{\frac{5}{\pi}}\sin\theta(1+\cos\theta)e^{i\phi} \quad (148)
$$

$$
{}_{-2}Y^{22} = \frac{1}{8}\sqrt{\frac{5}{\pi}}(1+\cos\theta)^2 e^{2i\phi} \quad (149)
$$

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