LIGO-T1300127: IMPROVING PERFORMANCE OF THE τ_0, τ_3 GENERATOR IN SBANK

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What we want to do is have a template bank where the templates can be chosen to be evenly distributed in $\tau_0 - \tau_3$ space, but where they are restricted to lines of constant mass ratio (or symmetric mass ratio η). We start by looking at how all these quantities are related.

First, some formulas (from `urand_tau0tau3_generator` in `tau0tau3.py`):

$$A_0 = \frac{5}{(256\pi f_{\text{low}})^{8/3}} \tag{1}$$

$$A_3 = \frac{\pi}{(8\pi f_{\text{low}})^{5/3}} \tag{2}$$

$$\tau_0 = \frac{A_0}{\eta M^{5/3}} \tag{3}$$

$$\tau_3 = \frac{A_3}{\eta M^{2/3}}, \tag{4}$$

where f_{low} is a minimum frequency cut-off, $\eta = \frac{m_1 m_2}{(m_1+m_2)^2}$ is the symmetric mass ratio, and $M = m_1 + m_2$ is the total mass.

Since the templates are evenly spaced in $\tau_0 - \tau_3$ space, $P(\tau_0, \tau_3) = \text{const}$. We do a change of coordinates to determine this probability distribution for M and η:

$$d\tau_0 d\tau_3 = \text{abs}\left(\det\begin{vmatrix} \frac{\partial\tau_0}{\partial\eta} & \frac{\partial\tau_0}{\partial M} \\ \frac{\partial\tau_3}{\partial\eta} & \frac{\partial\tau_3}{\partial M} \end{vmatrix}\right) dM d\eta \tag{5}$$

$$= \frac{A_0 A_3}{\eta^3 M^{10/3}} dM d\eta. \tag{6}$$

Thus we have:

$$P(\eta, M) = \frac{A_0 A_3}{\eta^3 M^{10/3}}. \tag{7}$$

We can immediately see that the following is true:

$$P(\eta, M) \propto P(\eta) P(M), \tag{8}$$

that is, $P(\eta, M)$ can be trivially factored into separate distributions of M and η.

$$P(\eta) \propto \eta^{-3} \tag{9}$$

$$P(M) \propto M^{-10/3} \tag{10}$$

Therefore, η and M can be chosen independently of each other. These are chosen via a random generator, but we must keep the above distributions in mind when choosing the values. Thus, we must transform an even distribution $x \in [0, 1)$ into distributions of $M \in [M_{\text{min}}, M_{\text{max}})$ and $\eta \in [\eta_{\text{min}}, \eta_{\text{max}})$. Let us first start with M. Since the probability densities are power laws, we can use the following relationship:

$$M = \alpha(x + \beta)^n, \tag{11}$$

where α, β, and n are constants. Using (D.1.2) of [1], we establish the following:

$$P(M) = |n^{-1}\alpha^{-\frac{1}{n}} M^{\frac{1}{n}-1}| P(x). \tag{12}$$

We get exactly the same relationship as in (D.1.2) of [1] because the β ends up cancelling out when you take the derivative of M with respect to x and put M back into the equation. In this case, $P(x)$ is a constant, and so, inserting the relationship in eq. 10 we get

$$M^{-\frac{10}{3}} \propto M^{-\frac{1}{n}-1}, \tag{13}$$

which gives us a value for $n = -\frac{3}{7}$. To find the values of the other constants in eq. 11, we look at the boundaries of each distribution. At the lower limit, we have $x = 0$ and $M = M_{\text{max}}$, which gives

$$M_{\text{max}} = \alpha\beta^n. \tag{14}$$

Likewise, the upper limit gives $x = 1$ at $M = M_{\text{min}}$, which leads to

$$M_{\text{min}} = \alpha(1 + \beta)^n. \tag{15}$$

Dividing these two equations and putting in the already-known value for n, we first obtain a value for β:

$$\beta = \left(\frac{M_{\text{min}}}{M_{\text{max}}}\right)^{\frac{7}{3}} \left[1 - \left(\frac{M_{\text{min}}}{M_{\text{max}}}\right)^{\frac{7}{3}}\right]^{-1}, \tag{16}$$

and putting this value back into one of the limit equations above, we obtain a value for α:

$$\alpha = M_{\text{min}} \left[1 - \left(\frac{M_{\text{min}}}{M_{\text{max}}}\right)^{\frac{7}{3}}\right]^{-\frac{3}{7}}. \tag{17}$$

This probability distribution can be tested if we choose some random even distribution in x and map it to the distribution we have just calculated for M. The

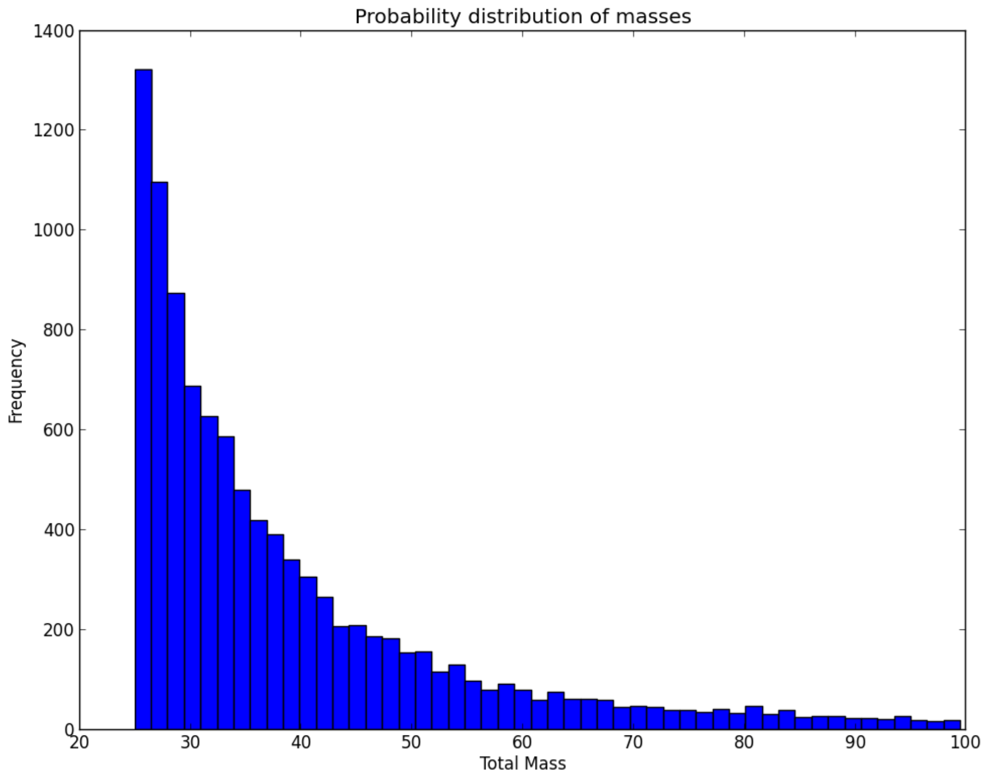

FIGURE 1. Histogram of distribution of total mass based on its relationship to an evenly-distributed set of random numbers between 0 and 1.

result of this can be seen in Fig. 1. Plotted here is the histogram for the distribution described in eq. 11 with $M_{\text{min}} = 25M_\odot$ and $M_{\text{max}} = 100M_\odot$. The histogram has the right minimum and maximum mass, at least, and a power law shape. To verify that the distribution has the correct power law exponent, we plot the histogram on logarithmic axes as shown in Fig. 2. In this graph, we have also plotted a power law with exponent -10/3 as the green curve. It is clear that the mass distribution follows the expected trend from eq. 10.

We can follow the same procedure for η, but this time with the relationship in eq. 9. Starting with this equation:

$$\eta = \gamma(y + \delta)^m, \tag{18}$$

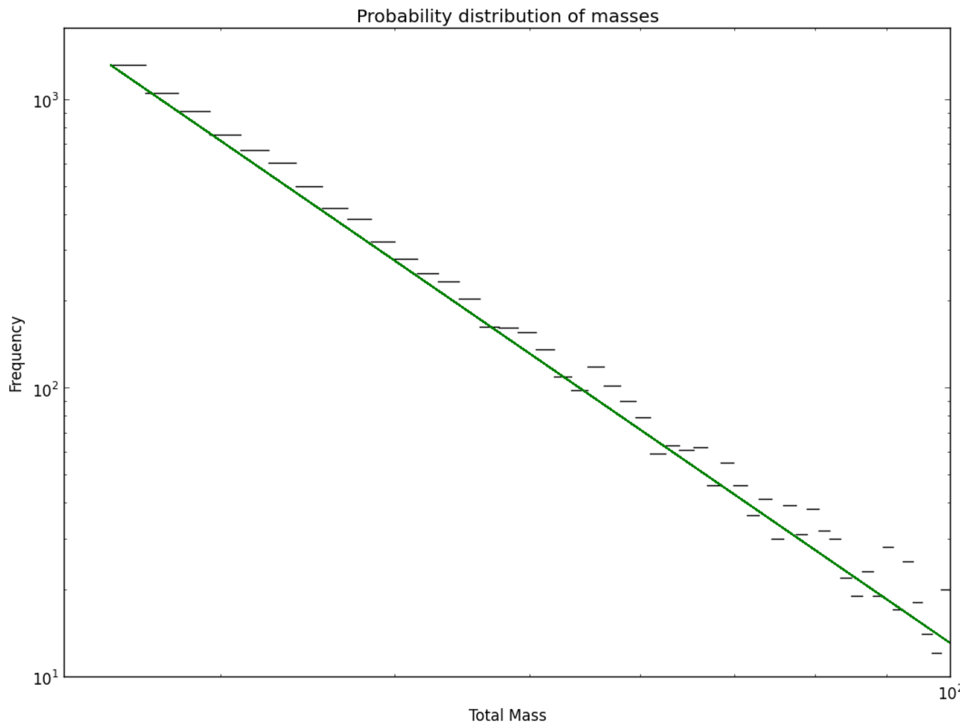

FIGURE 2. Log plot of the histogram shown in Fig. 1 over-plotted with a curve proportional to $M^{-10/3}$.

where γ, δ, and m are constants, and, like x, y is an even distribution over $[0, 1)$, we obtain $m = -\frac{1}{2}$,

$$\delta = \left(\frac{\eta_{\min}}{\eta_{\max}}\right)^2 \left[1 - \left(\frac{\eta_{\min}}{\eta_{\max}}\right)^2\right]^{-1}, \tag{19}$$

and

$$\gamma = \eta_{\min} \left[1 - \left(\frac{\eta_{\min}}{\eta_{\max}}\right)^2\right]^{-\frac{1}{2}}. \tag{20}$$

References

[1] Kipp Cannon, *Collapse-time distribution for large cosmic structures*. Ph.D dissertation, 2004.

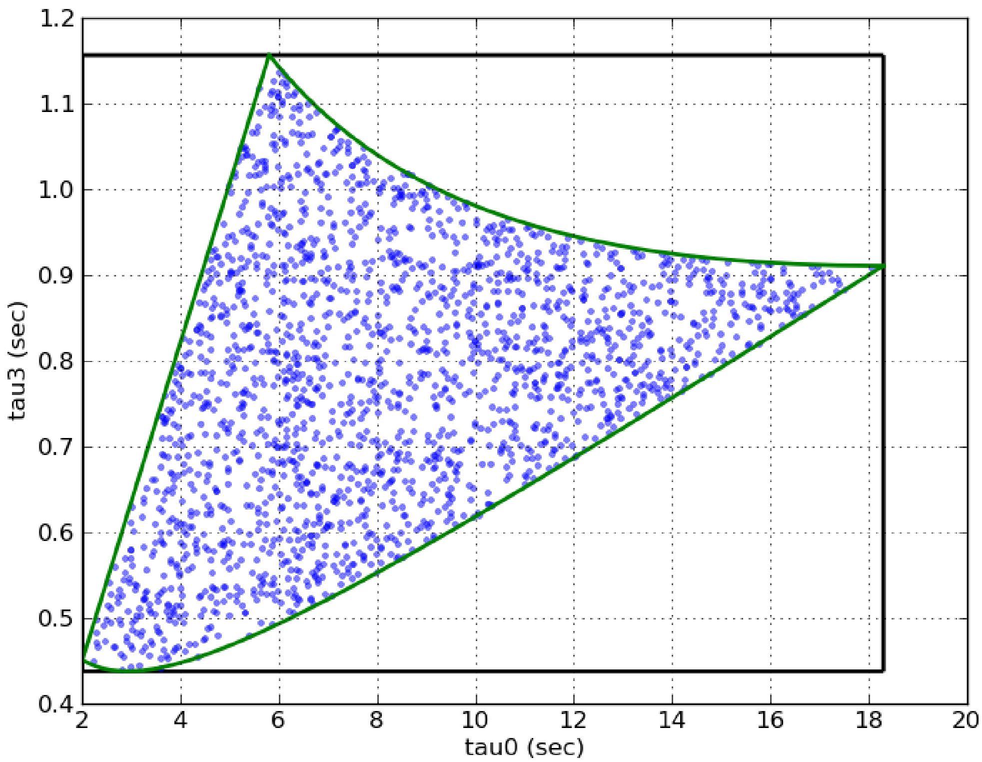

FIGURE 3. Scatter plot of $\tau_0 - \tau_3$ distribution based on the described distribution in $M - \eta$ space.