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On the accumulated round-trip Gouy phase shift for a general optical cavity

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1 Introduction

It is well known that the stability of a two-mirror Fabry-Perot cavity can be characterized by the product of two values called g-factors:

$$g_1 = 1 - \frac{L}{R_1} \tag{1}$$

$$g_2 = 1 - \frac{L}{R_2}, \tag{2}$$

where L is the length of the cavity, and R_1 and R_2 are the curvature radii of the cavity mirrors. When the product $g_1 g_2$ fulfills the following condition, the cavity has stable eigenmodes.

$$0 \leq g_1 g_2 \leq 1 \tag{3}$$

The product of the g-factors also provides a handy way to calculate the transverse mode spacing (TMS) of the cavity. The product of the g-factors is closely related to the accumulated round-trip Gouy phase shift (ζ) between the cavity eigenmodes[1].

$$\zeta = 2\cos^{-1} \pm\sqrt{g_1 g_2} \tag{4}$$

Here the sign $\pm$ is determined by the sign of g_1, and in fact g_1 and g_2 have the same sign for a stable cavity (cf. Eq. (3)). Consequently, the resulting TMS is given by the following formula

$$\nu_{\rm TMS} = \frac{\zeta}{2\pi}\nu_{\rm FSR} = \frac{\cos^{-1} \pm\sqrt{g_1 g_2}}{\pi}\nu_{\rm FSR} \tag{5}$$

where $\nu_{\rm FSR}$ is the free spectral range of the cavity that is given by $\nu_{\rm FSR} = c/(2L)$.

For a general cavity case (like a ring caity or a folded cavity with intra-cavity lensing optics), one would naturally imagine that there might be a similar simple indicator to the g-factors. In fact, J. Rollins recently conjectured from the discussion of the folded recycling cavity of the 40m prototype that ζ can be approximately calculated with the product of the generalized g-factors

$$\zeta = 2\cos^{-1}\sqrt{\prod_{i=1}^{n} g_i} \tag{6}$$

$$g_i = 1 - \frac{L_{\rm RT}}{2R_i} \tag{7}$$

where $L_{\rm RT}$ is the round-trip length of the cavity, and R_i is the curvature radius of the i-th mirror. However, **this conjecture is obviously not applicable to every cavity**; N. Smith pointed out that permutation of the mirrors in a ring cavity (like the output mode cleaner cavity) changes the TMS while the corresponding permutation of the g-factors in the

[1] The word "accumulated Gouy phase shift" means the shift of the optical phase between one of the Gaussian modes and the mode at the next higher-order spacial mode (e.g. $\rm TEM_{00}$ and $\rm TEM_{01}$) caused by Gouy phase effect. The "round-trip" one is the same quantity for a single round-trip of the optical cavity.

product does not change ζ. In fact, it will be shown in this document that **this conjecture is useful only when $1 - g_i \ll 1$ for all i.** [2]

As long as the simple and exact indicator like the g-factors is not available, we need to go back to the first principle of the beam calculation: ABCD matrix. If we calculate the ABCD matrix of the cascaded optical system by multiplying the individual ABCD matrices, all aspect of the system in terms of the beam parameter can be characterized. This can be understood by looking at the formula for Gaussian beam transformation by the ABCD matrix that is derived from Huygens propagarion integral ([1], Chapter 20):

$$q_{\rm out} = \frac{A q_{\rm in} + B}{C q_{\rm in} + D}, \tag{8}$$

where $q_{\rm in}$ and $q_{\rm out}$ are the q-parameters for the input and output beam respectively. A, B, C, and D are the elements of the ABCD matrix of the optical system. This formula means that different optical systems with the same ABCD matrix result in identical beam transformations.

It is also known that the quantity $(A+D)/2$ is closely related to cavity stability. The cavity eigenmodes are supposed to fulfill the beam-reproducing condition $q_{\rm out} = q_{\rm in}$. By solving this condition, Eq.(8) with the unitarity condition (i.e. $AD - BC = 1$), we obtain two solutions for the forward and backward beams:

$$q = \frac{A - D \pm \sqrt{(A+D)^2 - 4}}{2C} \tag{9}$$

As q is a complex number[3], the stability criteria is given by

$$-1 \le \frac{A+D}{2} \le 1. \tag{10}$$

Equivalently, this can be expressed as

$$0 \le \frac{A+D+2}{4} \le 1. \tag{11}$$

As we will find in Section 3, $\dfrac{A+D+2}{4}$ for a Fabry-Perot cavity corresponds to the product of the g-factors $g_1 g_2$. Therefore Eq. (11) is equivalent to Eq. (3).

In this note, the interpretation of Eqs. (10) and (11) are extended for general cavities. It is derived that **the accumulated round-trip Gouy phase shift can be computed only from the round-trip ABCD matrix of the cavity** as:

$$\zeta = {\rm sgn} B \cdot \cos^{-1}\left(\frac{A+D}{2}\right) \;, \tag{12}$$

where the value range of $\cos^{-1} x$ is defined to be

$$0 \le \cos^{-1} x < 2\pi \;\; (-1 < x \le 1). \tag{13}$$

[2]Therefore, the product of the generalized g-factors is always positive. There is no sign ambiguity.
[3]cf. $q = z + {\rm i} z_{\rm R}$, where $z_{\rm R}$ is the Rayleigh range.

Eq. (12) can equivalently be expressed as

$$\zeta = 2\cos^{-1}\left(\mathrm{sgn}B \cdot \sqrt{\frac{A+D+2}{4}}\right) \quad . \tag{14}$$

These expressions requires the calculation of the ABCD matrix for the round-trip path, and therefore more complicated to calculate than the product of the g-factors. However, it still provides concise evaluation of the stability on the general cavities.

2 Derivation

The derivation of Eqs. (12) and (14) is based on the following two points:

1. The accumulated Gouy phase shift between an arbitrary two points of a Gaussian beam can be computed only from the ABCD matrix between the points and the parameter of the input beam. Since the detail of this point is proven elsewhere [2], we will use this result without re-derivation.

2. The beam parameters of a cavity eigenmode id necessarily reproduced after a round trip in the cavity. Therefore, the cavity eigenmode must fulfill the beam-reproducing condition $q_{\mathrm{out}} = q_{\mathrm{in}}$. This means that the input and output beams have the same waist radii, same waist positions, and thus the same wavefront radii of curvature.

Erden et al. [2] described how the beam parameters (i.e. the beam radius, wave front radius of curvature, and Gouy phase) are described by the elements of the ABCD matrix and the parameters of the input beam Eqs. (13)∼(15) in [2]. Here are the excerpted equations rewritten:

$$\omega_{\mathrm{out}}^2 = \omega_{\mathrm{in}}^2\left(A + \frac{B}{r_{\mathrm{in}}}\right)^2 + \frac{B^2\lambda^2}{\pi^2\omega_{\mathrm{in}}^2} \tag{15}$$

$$\frac{1}{r_{\mathrm{out}}} = \frac{\left(C + \frac{D}{r_{\mathrm{in}}}\right)\left(A + \frac{B}{r_{\mathrm{in}}}\right) + \frac{BD\lambda^2}{\pi^2\omega_{\mathrm{in}}^4}}{\left(A + \frac{B}{r_{\mathrm{in}}}\right)^2 + \frac{B^2\lambda^2}{\pi^2\omega_{\mathrm{in}}^4}} \tag{16}$$

$$\tan\zeta = \frac{B}{\left(A + \frac{B}{r_{\mathrm{in}}}\right)\frac{\pi\omega_{\mathrm{in}}^2}{\lambda}} \tag{17}$$

Note that the ABCD matrix in [2] is defined differently from the equations above. The equations here have been modified so that the convention of the elements A, B, C, D agrees with the usual conventions in the optics book like [1]. (i.e. $B_{\mathrm{Siegman}} = B_{\mathrm{Erden}}/\lambda, C_{\mathrm{Siegman}} = C_{\mathrm{Erden}}\lambda$)

Indeed, the consequence of their paper, shown as Eq. (17) here, is a useful expression. This expression enables us to calculate the accumulated Gouy phase shift, ζ, from $r_{\rm in}$, $\omega_{\rm in}$ and the elements of the ABCD matrix, no matter what optical elements are in the path. This may be particularly useful for the calculation of ζ for the Gouy phase telescope of the wave front sensing (WFS) systems.

Erden et al. conversely derived the elements of the ABCD matrix from the beam parameters.

$$A = \frac{\omega_{\rm out}}{\omega_{\rm in}}\cos\zeta - \frac{B}{r_{\rm in}} \tag{18}$$

$$B = \frac{\pi\omega_{\rm in}\omega_{\rm out}}{\lambda}\sin\zeta \tag{19}$$

$$C = \frac{A}{r_{\rm out}} - \frac{\frac{1}{r_{\rm in}}\left(A + \frac{B}{r_{\rm in}}\right) + \frac{B\lambda^2}{\pi^2\omega_{\rm in}^4}}{\left(A + \frac{B}{r_{\rm in}}\right)^2 + \frac{B^2\lambda^2}{\pi^2\omega_{\rm in}^4}} \tag{20}$$

$$D = \frac{1+BC}{A} \tag{21}$$

The last equation comes from the unitarity of the ABCD matrix.

After simplification, the quantity $\frac{A+D}{2}$ can be expressed as follows

$$\frac{A+D}{2} = \frac{1}{2}\left(\frac{\omega_{\rm in}}{\omega_{\rm out}} + \frac{\omega_{\rm out}}{\omega_{\rm in}}\right)\cos\zeta + \frac{1}{2}\left(\frac{1}{r_{\rm out}} - \frac{1}{r_{\rm in}}\right)\frac{\pi\omega_{\rm in}\omega_{\rm out}}{\lambda}\sin\zeta \tag{22}$$

For the round-trip Gouy phase shift of a cavity, the formula is significantly simplified because of the beam-reproducing condition. We therefore find

$$\frac{A+D}{2} = \cos\zeta \tag{23}$$

and

$$\frac{A+D+2}{4} = \frac{\cos\zeta + 1}{2} = \cos^2\frac{\zeta}{2} \tag{24}$$

Because of the multi-value nature of the inverse cosine function there is a sign ambiguity:

$$\cos(|\zeta|) = \cos(-|\zeta|) \tag{25}$$

The key to solve this ambiguity is the element B. As the sign of B is determined by $\sin\zeta$, ${\rm sgn}B$ indicates the sign of ζ. Therefore we obtain the following expressions

$$\zeta = {\rm sgn}B \cdot \cos^{-1}\left(\frac{A+D}{2}\right)\ , \tag{26}$$

where the value range of $\cos^{-1}\zeta$ is

$$0 \le \cos^{-1}\zeta < 2\pi\ \ (-1 < \zeta \le 1). \tag{27}$$

If we use the inverse function of Eq. (24) for the value range above, we find $\cos\frac{\zeta}{2}$ is always positive and the ambiguity is hidden in the sign of the square-root term. Therefore the sign needs to be inside of the inverse cosine:

$$\zeta = 2\cos^{-1}\left(\mathrm{sgn}B \cdot \sqrt{\frac{A+D+2}{4}}\right) \quad . \tag{28}$$

3 Example 1: Fabry-Perot cavity

A two-mirror Fabry-Perot caity is chosen for the first example in order to confirm the well-known stability criteria using mirror g-factors.

The ABCD matrix for a space with the length of L:

$$\mathcal{S}(L) = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \tag{29}$$

The ABCD matrix for a mirror with the curvature radius of R:

$$\mathcal{F}(R) = \begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \tag{30}$$

Let's assume we have a linear cavity with the length of L and the curvature radii of the input and end mirrors as R_1 and R_2. The round-trip ABCD matrix is calculated as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathcal{F}(R_1)\,\mathcal{S}(L)\,\mathcal{F}(R_2)\,\mathcal{S}(L) \tag{31}$$

$$= \begin{pmatrix} 1-\dfrac{2L}{R_2} & 2L-\dfrac{2L^2}{R_2} \\ -\dfrac{2}{R_2}-\dfrac{2}{R_1}+\dfrac{4L}{R_1R_2} & 1-\dfrac{2L}{R_2}-\dfrac{4L}{R_1}+\dfrac{4L^2}{R_1R_2} \end{pmatrix} . \tag{32}$$

From this, we can calculate the cosine of the round-trip Gouy phase shift as

$$\cos\zeta = \frac{A+D}{2} \tag{33}$$

$$= 1-2\frac{L}{R_1}-2\frac{L}{R_2}+2\frac{L}{R_1}\frac{L}{R_2} \tag{34}$$

$$= 2\left(1-\frac{L}{R_1}\right)\left(1-\frac{L}{R_2}\right)-1 \tag{35}$$

$$\equiv 2g_1g_2-1 \tag{36}$$

or the quantity equivalent to the product of the g-factors

$$\cos^2\frac{\zeta}{2} = \frac{A+D+2}{4} \tag{37}$$

$$= \left(1-\frac{L}{R_1}\right)\left(1-\frac{L}{R_2}\right) \tag{38}$$

$$\equiv g_1g_2 \tag{39}$$

The sign of the g-factor can be extracted from the element B:

$$\mathrm{sgn} B = \mathrm{sgn}\left[2L\left(1-\frac{L}{R_2}\right)\right] \tag{40}$$

$$= \mathrm{sgn}\, g_2 \tag{41}$$

Therefore we obtain

$$\zeta = 2\cos^{-1}(\mathrm{sgn}\, g_2 \cdot \sqrt{g_1 g_2}). \tag{42}$$

This is equivalent to the famous expression Eq. (4).

4 Example 2: aLIGO OMC

The second example is the aLIGO output mode cleaner (OMC). The cavity of the OMC is a bow-tie style ring cavity formed by four mirrors: two flat input/output mirrors (FMs) and two curved mirrors (CMs) with the curvature radius of $R = 2.575$m (as the average number of measured values). The cavity mirrors are arranged in the order of FM1-FM2-CM1-CM2-FM1. The distances FM1-FM2 and CM1-CM2 are both $L_1 = 0.2816\,\mathrm{m}$, while the distances FM2-CM1, and CM2-FM1 are both $L_2 = 0.2844\,\mathrm{m}$. This means the round-trip length is $L_{\mathrm{RT}} = 1.132\,\mathrm{m}$, and the FSR is $\nu_{\mathrm{FSR}} = 264.8\,\mathrm{MHz}$. The angle of incidence on each mirror is $\theta = 4.042\,\mathrm{deg}\,(= 0.0706\,\mathrm{rad})$.

The round-trip ABCD matrix $\mathcal{C}$ is defined as

$$\mathcal{C}_\pm = \mathcal{M}_\pm\, \mathcal{S}(L_2)\mathcal{M}_\pm\, \mathcal{F}(R_\pm)\, \mathcal{S}(L_1)\mathcal{M}_\pm\, \mathcal{F}(R_\pm)\, \mathcal{S}(L_2)\mathcal{M}_\pm\, \mathcal{S}(L_1). \tag{43}$$

Because of the non-zero incident angle, the system naturally involves astigmatism. The signs $+$ and $-$ are applied to the vertical and horizontal mode, respectively. The reflection matrix $\mathcal{M}_\pm$ is defined as

$$\mathcal{M}_\pm = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \tag{44}$$

which correspond to the phase flip of the horizontal modes when the beam is reflected with non-zero incident angle in the horizontal plane. This matrix becomes important for an odd-number-mirror ring cavity, while we can ignore them for our four mirror cavity. The curved mirror therefore behaves as an astigmatic mirror with an effective radius of curvature of

$$R_\pm = R \cdot (\cos\theta)^{\pm 1} \ . \tag{45}$$

The results of the mode calculations are shown below.

Horizontal mode

$$\mathcal{C}_- = \begin{pmatrix} 0.3892 & 0.7242 \\ -1.381 & 0.0004690 \end{pmatrix} \tag{46}$$

Thus we obtain

$$\zeta = \mathrm{sgn} B\,.\, \cos^{-1}\frac{A+D}{2} = 1.375\,\mathrm{rad} \Rightarrow 78.76\ \mathrm{deg} \tag{47}$$

$$\nu_{\mathrm{TMS,H}} = \frac{\zeta}{2\pi}\,\nu_{\mathrm{FSR}} = 0.2188\,\nu_{\mathrm{FSR}} = 57.94\mathrm{MHz} \tag{48}$$

Vertical mode

$$\mathcal{C}_+ = \begin{pmatrix} 0.3864 & 0.7223 \\ -1.387 & -0.0004043 \end{pmatrix} \tag{49}$$

Thus we obtain

$$\zeta = \mathrm{sgn} B \cdot \cos^{-1} \frac{A+D}{2} = 1.378\,\mathrm{rad} \Rightarrow 78.98\ \mathrm{deg} \tag{50}$$

$$\nu_{\mathrm{TMS,V}} = \frac{\zeta}{2\pi} \nu_{\mathrm{FSR}} = 0.2194\,\nu_{\mathrm{FSR}} = 58.10\mathrm{MHz} \tag{51}$$

5 Example 3: Rollins conjecture

J. Rollins conjectured that the accumulated round-trip Gouy phase shift ζ of a multi-mirror cavity can be approximated by the product of the generalized g-factors g_i, as follows:

$$\zeta = 2\cos^{-1} \sqrt{\prod_{i=1}^{n} g_i} \tag{52}$$

$$g_i \equiv 1 - \frac{L_{\mathrm{RT}}}{2R_i} \tag{53}$$

where L_{RT} is the round-trip length of the cavity, and R_i is the curvature radius of the i-th mirror.

The first guess is that if all of $\delta g_i = 1 - g_i$ are small enough, $\dfrac{A+D+2}{4}$ may give us a good approximation of the the generalized g-factor product. Let's prove it in the following subsections.

In the following derivation it is convenient to define the ABCD matrix of a curved mirror and a space as

$$\mathcal{P}_i = \mathcal{F}(R_i)\,\mathcal{S}(L_i) = \begin{pmatrix} 1 & L_i \\ -\dfrac{2}{R_i} & 1 - \dfrac{2L_i}{R_i} \end{pmatrix} = \begin{pmatrix} 1 & L_i \\ -\dfrac{\delta f_i}{L_{\mathrm{RT}}} & 1 - \dfrac{\delta f_i}{L_{\mathrm{RT}}} L_i \end{pmatrix} \tag{54}$$

where

$$\delta f_i \equiv \frac{2L_{\mathrm{RT}}}{R_i} = 4\delta g_i \tag{55}$$

Proposition:

The ABCD matrix of the optical system $\mathcal{P}_n \mathcal{P}_{n-1} \ldots \mathcal{P}_1$ is represented in the following form for $n \geq 2$, up to the first order of δf_i.

$$\mathcal{P}_n \mathcal{P}_{n-1} \ldots \mathcal{P}_1 = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}, \tag{56}$$

where

$$A_n = 1 - \sum_{i=2}^{n} \left(L_i \sum_{j=1}^{i-1} \frac{\delta f_j}{L_{\mathrm{RT}}} \right) \tag{57}$$

$$B_n = \sum_{i=1}^{n} L_i - \sum_{i=1}^{n-1} \left[\frac{\delta f_i}{L_{\mathrm{RT}}} \left(\sum_{j=1}^{i} L_j \right) \left(\sum_{k=i+1}^{n} L_k \right) \right] \tag{58}$$

$$C_n = - \sum_{i=1}^{n} \frac{\delta f_i}{L_{\mathrm{RT}}} \tag{59}$$

$$D_n = 1 - \sum_{i=1}^{n} \left(\frac{\delta f_i}{L_{\mathrm{RT}}} \sum_{j=1}^{i} L_j \right) \tag{60}$$

Proof:

For $n = 2$, we directly calculate $\mathcal{P}_2\mathcal{P}_1$:

$$\mathcal{P}_2\mathcal{P}_1 = \begin{pmatrix} 1 & L_2 \\ -\frac{\delta f_2}{L_{\mathrm{RT}}} & 1 - \frac{\delta f_2 L_2}{L_{\mathrm{RT}}} \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ -\frac{\delta f_1}{L_{\mathrm{RT}}} & 1 - \frac{\delta f_1 L_1}{L_{\mathrm{RT}}} \end{pmatrix} \tag{61}$$

$$= \begin{pmatrix} 1 - L_2 \frac{\delta f_1}{L_{\mathrm{RT}}} & L_1 + L_2 - \frac{\delta f_1}{L_{\mathrm{RT}}} L_1 L_2 \\ -\frac{\delta f_1 + \delta f_2}{L_{\mathrm{RT}}} & 1 - \frac{\delta f_1}{L_{\mathrm{RT}}} L_1 - \frac{\delta f_2}{L_{\mathrm{RT}}} (L_1 + L_2) \end{pmatrix} + \mathcal{O}^2(\delta f_i) \tag{62}$$

Assuming the proposition is fulfilled for A_n, B_n, C_n and D_n, we calculate $\mathcal{P}_{n+1}\mathcal{P}_n \ldots \mathcal{P}_1$ as

$$\mathcal{P}_{n+1}\mathcal{P}_n \ldots \mathcal{P}_1 = \begin{pmatrix} 1 & L_{n+1} \\ -\frac{\delta f_{n+1}}{L_{\mathrm{RT}}} & 1 - \frac{\delta f_{n+1} L_{n+1}}{L_{\mathrm{RT}}} \end{pmatrix} \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} + \mathcal{O}^2(\delta f_i) \tag{63}$$

$$= \begin{pmatrix} A_n - L_{n+1} \sum_{i=1}^{n} \frac{\delta f_i}{L_{\mathrm{RT}}} & B_n + L_{n+1} \left[1 - \sum_{i=1}^{n} \left(L_i \sum_{j=i}^{n} \frac{\delta f_j}{L_{\mathrm{RT}}} \right) \right] \\ C_n - \frac{\delta f_{n+1}}{L_{\mathrm{RT}}} & D_n - \frac{\delta f_{n+1}}{L_{\mathrm{RT}}} \sum_{i=1}^{n} L_i \end{pmatrix} + \mathcal{O}^2(\delta f_i) \tag{64}$$

$$= \begin{pmatrix} A_{n+1} & B_{n+1} \\ C_{n+1} & D_{n+1} \end{pmatrix} + \mathcal{O}^2(\delta f_i) \tag{65}$$

Q.E.D.

Consequence

From Eqs. (57) and (60), we obtain

$$\frac{A+D+2}{4} \ (= \cos^2\frac{\zeta}{2}) \tag{66}$$

$$= 1 - \sum_{i=1}^{n}\left[\frac{\delta f_i}{4L_{\mathrm{RT}}}\left(\sum_{j=1}^{n} L_i\right)\right] + O^2(\delta f_i) \tag{67}$$

$$= 1 - \sum_{i=1}^{n}\frac{\delta f_i}{4} + O^2(\delta f_i) \tag{68}$$

$$= 1 - \sum_{i=1}^{n}\frac{L_{\mathrm{RT}}}{2R_i} + O^2(\delta f_i) \tag{69}$$

This quantity can be compared with the product of the generalized g-factors.

$$\prod_{i=1}^{n} g_i = \prod_{i=1}^{n}\left(1 - \frac{L_{\mathrm{RT}}}{2R_i}\right) \tag{70}$$

$$= \prod_{i=1}^{n}\left(1 - \frac{\delta f_i}{4}\right) \tag{71}$$

$$= 1 - \sum_{i=1}^{n}\frac{\delta f_i}{4} + O^2(\delta f_i) \tag{72}$$

$$= 1 - \sum_{i=1}^{n}\frac{L_{\mathrm{RT}}}{2R_i} + O^2(\delta f_i) \tag{73}$$

This agrees with Eq. (69).

This means that **Rollins conjecture is true if all of the curvature radii of the mirrors are long enough relative to the round-trip length of the cavity** (i.e. $\delta g_i \ll 1$ for all i). Note that this result means that permutation of the mirrors does not change the resulting round-trip Gouy phase significantly up to the first order in this small δg_i regime. Also note that all of the g-factors are kept in the positive number. So there is no ambiguity of the Gouy phase shift.

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