Approximation Methods for Bayesian Detection Statistics in a Targeted Search for Continuous Gravitational Waves

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Signal Model for Continuous GWs

Signal received at a detector from a nearly periodic source (e.g., spinning neutron star, white dwarf binary):

$$h(t) = F_+ A_+ \cos[\phi(t) + \phi_0] + F_\times A_\times \sin[\phi(t) + \phi_0] \qquad (1)$$

Orientation of angular momentum of system described by angles ι (inclination to line of sight) and ψ (angle on sky of projected angular momentum). Polarization amplitudes from signal amplitude h_0 & inclination ι:

$$A_+ = \frac{h_0}{2}(1+\cos^2\iota) = h_0\frac{1+\chi^2}{2} \qquad (2a)$$
$$A_\times = h_0 \cos\iota = h_0\chi \qquad (2b)$$

Antenna patterns $F_{+,\times}$ determined by polarization angle ψ & amplitude modulation coefficients a & b (which come from detector geometry & sky position as shown in figure 1):

$$F_+ = a\cos 2\psi + b\sin 2\psi \qquad (3a)$$
$$F_\times = -a\sin 2\psi + b\cos 2\psi \qquad (3b)$$

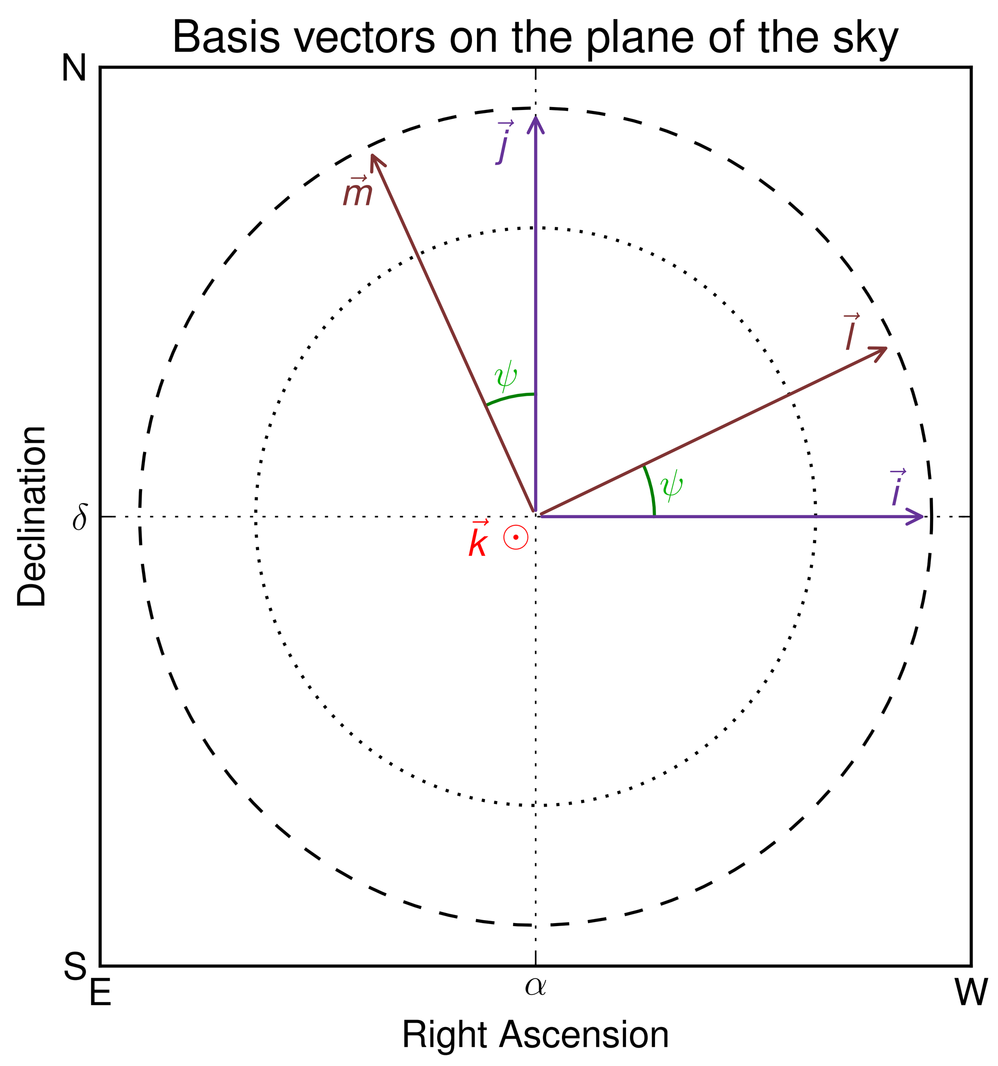

Figure 1: *Basis vectors which determine AM coefficients & antenna patterns from detector geometry tensor $\overleftrightarrow{d}$. Given sky position ⇒ propagation direction $\vec{k}$, can define $\vec{i}$ & $\vec{j}$ pointing "West & North on the sky" & construct basis tensors $\overleftrightarrow{\varepsilon}_+ = \vec{i}\otimes\vec{i} - \vec{j}\otimes\vec{j}$ & $\overleftrightarrow{\varepsilon}_\times = \vec{i}\otimes\vec{j} + \vec{j}\otimes\vec{i}$. AM coefficients are $a = \overleftrightarrow{d} : \overleftrightarrow{\varepsilon}_+$ & $b = \overleftrightarrow{d} : \overleftrightarrow{\varepsilon}_\times$. Preferred polarization basis aligns $\vec{l}$ or $\vec{m}$ along projected angular momentum of source (chosen so $-\pi/4 \le \psi \le \pi/4$) and defines $\overleftrightarrow{e}_+ = \vec{l}\otimes\vec{l} - \vec{m}\otimes\vec{m}$ & $\overleftrightarrow{e}_\times = \vec{l}\otimes\vec{m} + \vec{m}\otimes\vec{l}$. Antenna patterns are $F_{+,\times} = \overleftrightarrow{d} : \overleftrightarrow{e}_{+,\times}$.*

Divide signal parameters into

- *Amplitude parameters* $\mathcal{A} \equiv \{h_0, \chi = \cos\iota, \psi, \phi_0\}$
- *Phase parameters* $\lambda \equiv \{\alpha, \delta, f_0, f_1, \ldots\}$ which determine (Doppler modulated) $\phi(t)$

Detection Statistics: $\mathcal{F}$-stat & $\mathcal{B}$-stat

- Signal hypothesis $\mathcal{H}_s(\mathcal{A}_s, \lambda_s)$: $x(t) = n(t) + h(t; \mathcal{A}, \lambda)$
- Noise hypothesis $\mathcal{H}_n$: $x(t) = n(t)$

If signal parameters $\{\mathcal{A}_s, \lambda_s\}$ known, optimal detection statistic is likelihood ratio

$$\frac{P(x|\mathcal{H}_s(\mathcal{A}_s,\lambda_s))}{P(x|\mathcal{H}_n)} = \exp[\Lambda(x; \mathcal{A}_s, \lambda_s)] \qquad (4)$$

For targeted search, phase params λ_s (sky position, frequency, spindowns) known, but amp params $\mathcal{A}_s$ unknown. $\mathcal{F}$-statistic method [1] defines maximized log-likelihood ratio

$$\mathcal{F}(x) = \max_{\mathcal{A}} \ln\frac{P(x|\mathcal{H}_s(\mathcal{A},\lambda_s))}{P(x|\mathcal{H}_n)} = \max_{\mathcal{A}} \Lambda(x; \mathcal{A}, \lambda_s) \qquad (5)$$

Optimal statistic is actually $\mathcal{B}$-statistic [2] (Bayes factor; marginalized, not maximized)

$$\mathcal{B}(x) = \frac{P(x|\mathcal{H}_s)}{P(x|\mathcal{H}_n)} = \frac{\int d\mathcal{A}\, P(x|\mathcal{H}_s(\mathcal{A},\lambda_s))\, P(\mathcal{A}|\mathcal{H}_s(\lambda_s))}{P(x|\mathcal{H}_n)} = \int d\mathcal{A} \exp[\Lambda(x; \mathcal{A}, \lambda_s)] \qquad (6)$$

New Coordinates on Amplitude Parameter Space

[1] introduce functions $\{\mathcal{A}^\mu(h_0, \chi, \psi, \phi_0) | \mu = 1, \ldots 4\}$ so that

$$h(t) = \mathcal{A}^\mu h_\mu(t; \lambda) \quad (\text{implicit } \textstyle\sum_{\mu=1}^4) \qquad (7)$$

and $\Lambda(x; \mathcal{A}, \lambda)$ is quadratic in $\{\mathcal{A}^\mu\}$, allowing analytic maximization. We define a different set of such coordinates $\{\mathcal{A}^{\hat\mu}\}$ which are closer to the physical parameters:

$$\mathcal{A}^{\hat1} = \mathcal{P}^1 = p\cos\theta_p \quad \text{and} \quad \mathcal{A}^{\hat2} = \mathcal{P}^2 = p\sin\theta_p \qquad (8a)$$
$$\mathcal{A}^{\hat3} = \mathcal{Q}^1 = q\cos\theta_q \quad \text{and} \quad \mathcal{A}^{\hat4} = \mathcal{Q}^2 = q\sin\theta_q\,; \qquad (8b)$$

with

$$p = \frac{A_+ + A_\times}{2} = h_0\left(\frac{1+\chi}{2}\right)^2 \quad \text{and} \quad \theta_p = 2\psi + \phi_0\,; \qquad (9a)$$
$$q = \frac{A_+ - A_\times}{2} = h_0\left(\frac{1-\chi}{2}\right)^2 \quad \text{and} \quad \theta_q = 2\psi - \phi_0 \qquad (9b)$$

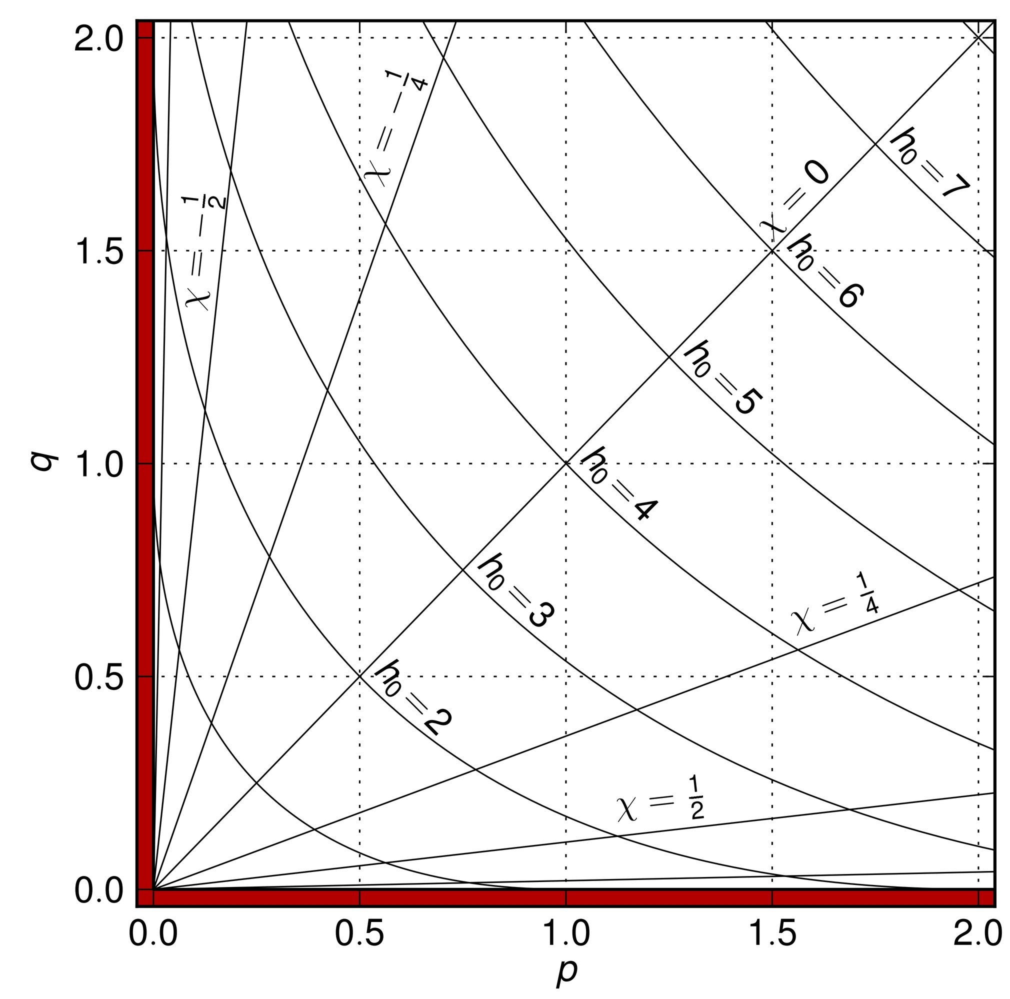

Figure 2: *Correspondence between radial polar coordinates p & q and physical amp params h_0 & $\chi = \cos\iota$. We plot lines of constant $h_0 \in [0,\infty)$ & $\chi \in [-1,1]$, drawn in first quadrant of the $\{p,q\}$ plane. (The red shaded represents unphysical coordinate values.) Circular polarization, $\chi = \pm1$, corresponds to $q = 0$ or $p = 0$.*

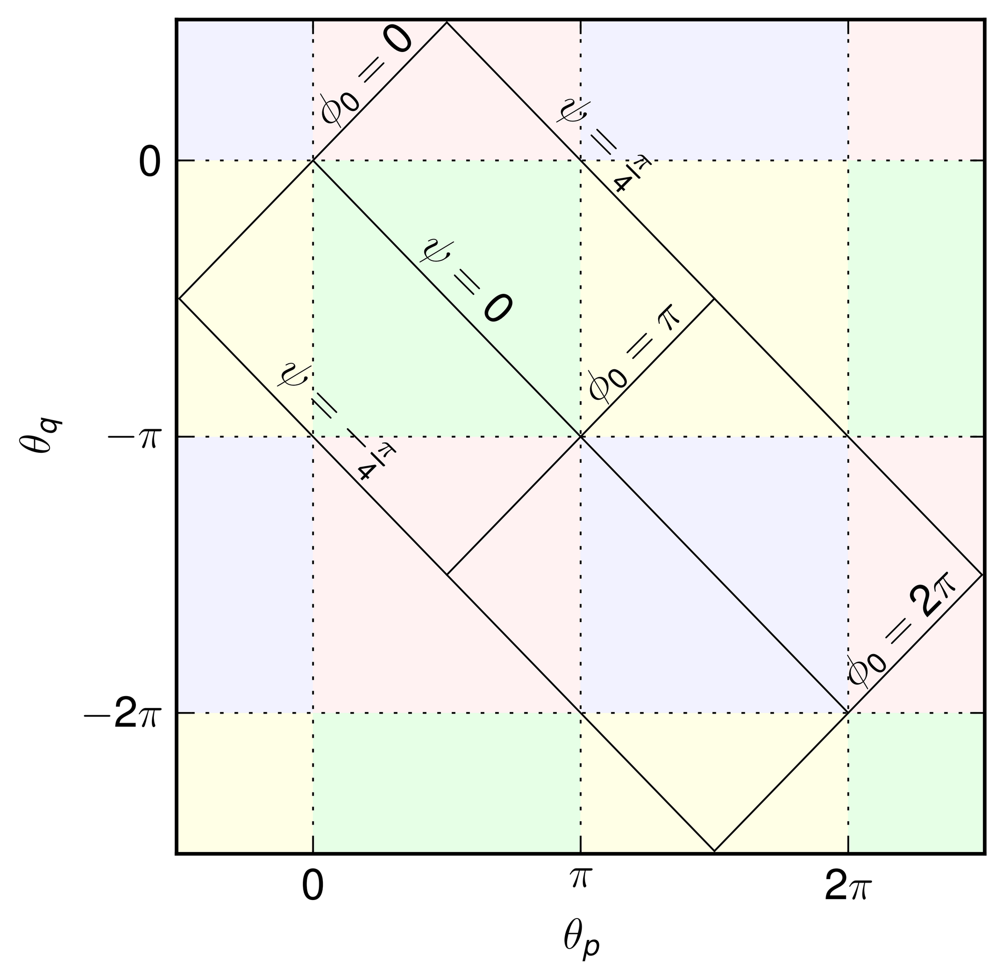

Figure 3: *Correspondence between angular polar coordinates θ_p & θ_q and physical amp params ϕ_0 & ψ. The principal region of polarization $\psi \in (-\pi/4, \pi/4]$ and phase $\phi_0 \in [0, 2\pi)$ is shown in the $\{\theta_p, \theta_q\}$ plane; θ_p and θ_q are each periodically identified, with period 2π. Note that since the transformation $\{\psi, \phi_0\} \to \{\psi + \pi/2, \phi_0 + \pi\}$ leaves the waveform unchanged, the edge $\psi = -\pi/4$, $\phi_0 \in [0,\pi)$ is actually identified with $\psi = \pi/4$, $\phi_0 \in [\pi, 2\pi)$, while $\psi = -\pi/4$, $\phi_0 \in [\pi, 2\pi)$ is identified with $\psi = \pi/4$, $\phi_0 \in [0,\pi)$. These periodic identifications show that the principal $\{\psi, \phi_0\}$ region is equivalent to the region $\theta_p \in [0, 2\pi)$, $\theta_q \in [0, 2\pi)$.*

In these coordinates, the log-likelihood is

$$\Lambda(x; \mathcal{A}, \lambda) = \mathcal{A}^{\hat\mu} x_{\hat\mu}(\lambda) - \frac{1}{2}\mathcal{A}^{\hat\mu}\mathcal{M}_{\hat\mu\hat\nu}(\lambda)\mathcal{A}^{\hat\nu}$$
$$= p\,(x_1\cos\theta_p + x_2\sin\theta_p) + q\,(x_3\cos\theta_q + x_4\sin\theta_q)$$
$$-\frac{1}{2}Ip^2 - \frac{1}{2}Jq^2 - pq\left[K\sin(\theta_p+\theta_q) + L\cos(\theta_p+\theta_q)\right] \qquad (10)$$

where $I = J$ in the long-wavelength limit.

Failure of the Gaussian Approximation

Given an unphysical prior $P(\{\mathcal{A}^{\hat\mu}\}|\mathcal{H}_s(\lambda_s)) = \text{const}$, the $\mathcal{B}$-statistic is equivalent to the $\mathcal{F}$-statistic [2]

$$\mathcal{B}_{\text{unphys}}(x; \lambda_s) \propto \int \exp[\Lambda(x; \mathcal{A}, \lambda_s)]\, d\mathcal{P}^1\, d\mathcal{P}^2\, d\mathcal{Q}^1\, d\mathcal{Q}^2 \propto e^{\mathcal{F}(x;\lambda_s)} \qquad (11)$$

because the Gaussian integral picks out the maximum likelihood point $\mathcal{A} = \widehat{\mathcal{A}}$. On the other hand, the physical (isotropic) prior is uniform in χ, ϕ_0 & ψ. For simplicity assume it's also uniform in h_0. Coordinate transforms show

$$dp\,dq = h_0\frac{1-\chi^2}{4}\,dh_0\,d\chi \quad \text{and} \quad d\theta_p\,d\theta_q = 4\,d\psi\,d\phi_0 \qquad (12)$$

so

$$d\mathcal{P}^1\, d\mathcal{P}^2\, d\mathcal{Q}^1\, d\mathcal{Q}^2 = 4\left(h_0\frac{1-\chi^2}{4}\right)^3 dh_0\,d\chi\,d\psi\,d\phi \qquad (13)$$

where we use $pq = \left(h_0\frac{1-\chi^2}{4}\right)^2$. This means, if we use isotropic priors, we get

$$\mathcal{B}_{\text{phys}}(x; \lambda_s) \propto \int \exp[\Lambda(x; \mathcal{A}, \lambda_s)]\, dh_0\, d\chi\, d\psi\, d\phi_0$$
$$\propto \int \frac{\exp[\Lambda(x; \mathcal{A}, \lambda_s)]}{\mathcal{J}(\mathcal{A})}\, d\mathcal{P}^1\, d\mathcal{P}^2\, d\mathcal{Q}^1\, d\mathcal{Q}^2 \qquad (14)$$

with Jacobian $\mathcal{J} \propto (pq)^{3/2}$. It's tempting to Taylor expand $\alpha(\mathcal{A}) = -\ln\mathcal{J}(\mathcal{A})$ about the maximum likelihood point:

$$\alpha(\mathcal{A}) = \widehat\alpha + \widehat\alpha_{\hat\mu}\Delta\mathcal{A}^{\hat\mu} + \frac{1}{2}\widehat\alpha_{\hat\mu\hat\nu}\Delta\mathcal{A}^{\hat\mu}\Delta\mathcal{A}^{\hat\nu} + \mathcal{O}([\Delta\mathcal{A}]^3) \qquad (15)$$

Then

$$\Lambda(x; \mathcal{A}, \lambda_s) + \alpha(\mathcal{A}) \approx -\frac{1}{2}\mathcal{N}_{\hat\mu\hat\nu}(x; \lambda_s)\Delta\mathcal{A}^{\hat\mu}\Delta\mathcal{A}^{\hat\nu} + \widehat\alpha_{\hat\mu}\Delta\mathcal{A}^{\hat\mu} + \widehat\alpha + \mathcal{F}(x) \qquad (16)$$

and we can approximate the $\{\mathcal{A}^{\hat\mu}\}$ integral as Gaussian. But this only works if $\mathcal{N}_{\hat\mu\hat\nu}(x; \lambda_s) = \mathcal{M}_{\hat\mu\hat\nu}(\lambda_s) - \widehat\alpha_{\hat\mu\hat\nu}(x; \lambda_s)$ is positive definite. In these coordinates, it's easy to calculate

$$\widehat\alpha_{\hat\mu\hat\nu} = \frac{3}{2}\begin{pmatrix} \frac{\cos 2\widehat\theta_p}{\widehat{p}^2} & \frac{\sin 2\widehat\theta_p}{\widehat{p}^2} & 0 & 0 \\ \frac{\sin 2\widehat\theta_p}{\widehat{p}^2} & -\frac{\cos 2\widehat\theta_p}{\widehat{p}^2} & 0 & 0 \\ 0 & 0 & \frac{\cos 2\widehat\theta_q}{\widehat{q}^2} & \frac{\sin 2\widehat\theta_q}{\widehat{q}^2} \\ 0 & 0 & \frac{\sin 2\widehat\theta_q}{\widehat{q}^2} & -\frac{\cos 2\widehat\theta_q}{\widehat{q}^2} \end{pmatrix} \qquad (17)$$

If ML params are close to circular polarization ($\widehat{p}$ or $\widehat{q}$ small), two of eigenvalues of $\mathcal{N}_{\hat\mu\hat\nu}(x; \lambda_s)$ will be $\pm\frac{3}{2\widehat{p}^2} \Longrightarrow$ not positive definite. Λ has a saddle point, not a ≈Gaussian peak.

Integration in physical parameter space

Examination of (10) shows that, since $p, q \propto h_0$ and $\theta_p + \theta_q = 4\psi$, the log-likelihood has tractable h_0 and ϕ_0 dependence:

$$\Lambda(x; \mathcal{A}) = h_0\,\omega(x; \chi, \psi)\cos[\phi_0 - \varphi_0(x; \chi, \psi)] - \frac{1}{2}h_0{}^2[\gamma(\chi, \psi)]^2 \qquad (18)$$

the h_0 & ϕ_0 can be done analytically to give

$$\mathcal{B} \propto \int_{-1}^{1} d\chi \int_{-\pi/4}^{\pi/4} d\psi\, \frac{I_0(\xi(x; \chi, \psi))\, e^{\xi(x;\chi,\psi)}}{\gamma(\chi, \psi)} \qquad (19)$$

where

$$\xi(x; \chi, \psi) = \frac{[\omega(x; \chi, \psi)]^2}{4[\gamma(\chi, \psi)]^2} \qquad (20)$$

Which leaves a 2D numerical integral. Since the ψ dependence is mostly oscillatory, we replace parts of the integrand with ψ-averaged versions:

$$\mathcal{B} \sim \int_{-1}^{1} d\chi\, \frac{I_0(\bar{\bar\xi}(x; \chi))\, e^{\bar{\bar\xi}(x;\chi)}}{\bar\gamma(\chi)} \qquad (21)$$

where $\bar{f} = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} d\psi\, f(\psi)$ and $\bar{\bar\xi}(x; \chi) = \frac{[\bar\omega(x;\chi)]^2}{4[\bar\gamma(\chi)]^2}$; we then only have to integrate numerically over χ. This statistic is still more powerful than the $\mathcal{F}$-statistic, but quicker to calculate than the exact $\mathcal{B}$-statistic. [3]

References

[1] Jaranowski, Królak & Schutz, *PRD* **58**, 063001 (1998)
[2] Prix & Krishnan, *CQG* **26**, 204013 (2009)
[3] Prix, Whelan & Cutler in progress