# High Fidelity Initial Models for Neutron Star Simulations

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# Outline

1. Motivation
   - Neutron Stars as Gravitational Wave Sources
   - Goals
2. Background
   - Numerical Methods
   - Initial Data Solvers
3. Results
   - Quantitative Methods of Comparison

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# Neutron Star Binaries

- Neutron star binaries (with other neutron stars, white dwarfs, or black holes) are a source of readily detectable gravitational waves.
- By doing time-evolution simulations of NS binaries, we can get precision models of the GW emissions.
- Such simulations require high precision initial models, as time evolution code tends to introduce additional numerical error.

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# Improvement Over Current Models

- Our current initial data solver, CST[1] (Cook-Shapiro-Teukolsky), is of limited accuracy. This limits the accuracy we can achieve with the time-evolution code.
- We require more accuracy as the current accuracy of our code limits the ability to which we can test and improve the time-evolution code.
- As we desire more accurate models than generated by CST, we must use a different initial data solver.
- We have focused on two initial data solvers in particular, AKM and LORENE.

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# Spectral Methods

- Spectral methods are a method of solving differential equations by approximating functions with high order polynomials on a small number of domains.
- The approximation is done by choosing an interpolating polynomial $I(x)$ which is equal to the original function $f(x)$ on some set of grid points, usually taken to be the extrema or roots of Chebyshev polynomials.
- Generally, the approximation is done in a domain $\tilde{x} \in [-1, 1]$ or $\tilde{x} \in [0, 1]$ which maps to some subset of $\mathbb{R}$.
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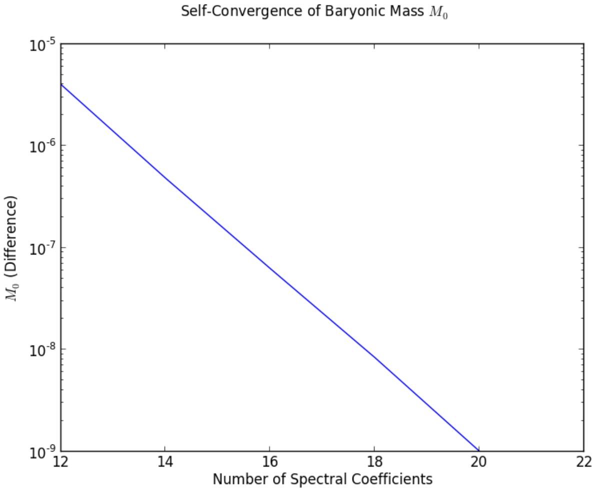
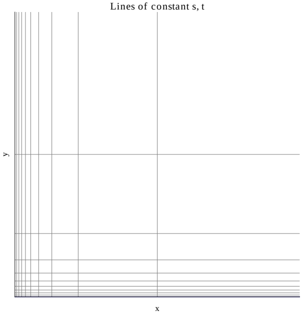
# Convergence of Spectral Methods



# Compactification

- We cannot use Cartesian coordinates in a numerical method if we wish to model the infinite domain $x \in [0, \infty)$.
- A change of coordinates such as $x = \frac{1-s}{s}$ maps the finite domain $[0, 1)$ to the infinite domain $[0, \infty)$.
- Compactification allows us to use the more natural boundary conditions at infinity.



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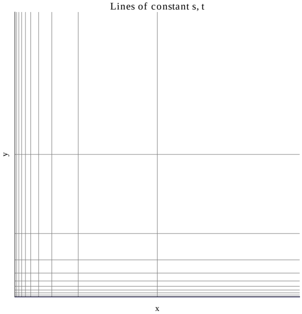

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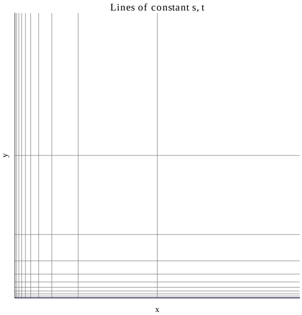

# Numerical Relativity

- We consider the case of an axisymmetric star, where the metric $ds^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta$ can be written in terms of four metric potentials which are functions of (in spherical coordinates) $r$ and $\theta$ only (in cylindrical coordinates, $\rho$ and $\zeta$).
- The star is generally considered to be a perfect fluid, with stress-energy tensor given by $T_{\alpha\beta} = (\varepsilon + p)\, u_\alpha u_\beta + p g_{\alpha\beta}$. Here $p$ is the pressure, $u_\alpha$ is the matter 4-velocity, and $\varepsilon$ is the total energy density.
- The various numerical schemes solve the GR and hydrodynamic equations for the metric potentials, density, and fluid velocity at every grid point inside the star, and for the metric potentials at every grid point outside the star, as well as the shape of the stellar surface.

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# AKM

- AKM is an initial data solver written by Markus Ansorg, Andreas Kleinwächter, and Reinhard Meinel[2].
- Physically, AKM models a rotating star as a rigidly rotating, axisymmetric, perfect fluid.
- AKM uses a 2-domain pseudo-spectral method, with the inner sub-domain corresponding to the stellar interior and a compactified external sub-domain.

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# LORENE

- LORENE is the Langage Objet pour la RElativité NumériquE[3], a C/C++ library of classes useful for numerical relativity and solving partial differential equations using multi-domain spectral methods.
- LORENE contains code that solves common astrophysical scenarios such as rotating stars, black holes, and binary systems.

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# GRV2

## Definition

In axisymmetric, asymptotically flat spacetime, with a metric of the form

$$g_{\alpha\beta}\mathrm{d}x^{\alpha}\,\mathrm{d}x^{\beta} = -e^{2\nu}\mathrm{d}t^2 + e^{2\mu}\left(\mathrm{d}\rho^2 + \mathrm{d}\zeta^2\right) + W^2 e^{-2\nu}\left(\mathrm{d}\varphi - \omega\mathrm{d}t\right)^2$$

the following virial identity holds:

$$\left| 1 - 8\pi \frac{\int_0^\infty \int_0^\infty \left( p + (\varepsilon + p)\frac{v^2}{1-v^2} \right) e^{2\mu}\,\mathrm{d}\rho\,\mathrm{d}\zeta}{\int_0^\infty \int_0^\infty \left( (\nabla\nu)^2 - \frac{3}{4} W^2 e^{-4\nu} (\nabla\omega)^2 \right) \mathrm{d}\rho\,\mathrm{d}\zeta} \right| = 0$$

where $(\nu, \omega, \mu, W)$ are the metric potentials, $p$ is the pressure, $\varepsilon$ is the total energy density, $v$ is the proper velocity, and $(t, \rho, \zeta, \varphi)$ are the (cylindrical) coordinates.

# GRV2 (Continued)

- In the Newtonian limit, GRV2 becomes $\int_0^\pi \int_0^\infty \left[p + \rho v^2 - \frac{1}{8\pi G} (\nabla \nu)^2\right] r \, \mathrm{d}r \, \mathrm{d}\theta = 0$, where $\frac{1}{8\pi G} (\nabla \nu)^2$ is the gravitational potential energy density. This roughly corresponds to $\int p \, \mathrm{d}V + mv^2 - U_g = 0$
- In an numerical method, GRV2 will not vanish entirely; it will converge to zero as the resolution increases.
- We can compare different numerical methods by examining how accurately they compute GRV2.

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# Computing GRV2

- The codes for rotating stars in LORENE already compute GRV2.
- Calculations of GRV2 in LORENE converge to order $10^{-7}$ (with 33 spectral coefficients in each of the internal domains).
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# AKM Spectral Domain

## Definition

In the outer domain,

$$\rho^2(s,t) = t\left[r_e^2 - r_p^2 + \left(r_p + r_e\frac{1-s}{s}\right)^2\right]$$

$$\zeta^2(s,t) = (1-t)\left[\left(r_p + r_e\frac{1-s}{s}\right)^2 - r_p^2\right] + G(t) - r_e^2 t$$

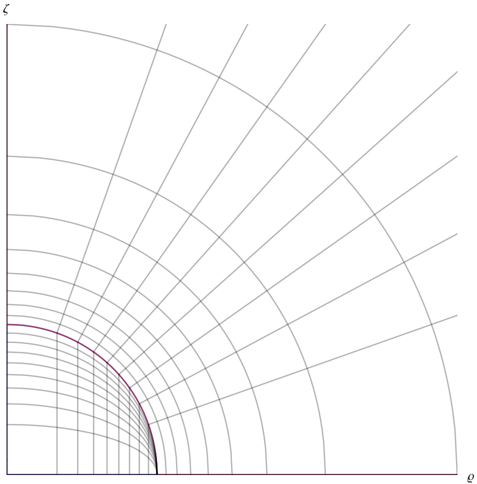
In the inner domain,

$$\rho^2(s,t) = r_e^2 t \qquad \zeta^2(s,t) = s\left[G(t) - r_e^2 t\right]$$

where $r_e$ is the equatorial radius, $r_p$ is the polar radius, $(s,t) \in [0,1]^2$ are the spectral coordinates, and $G(t) : [0,1] \to \mathbb{R}$ describes the stellar surface.

# AKM Spectral Domain

Lines of constant s, t



# Summary

- Accurate, high precision simulations of neutron star binaries are useful in studying their GW emissions; such models require precise initial models.
- Our initial models are generated by initial data solvers which solve the GR and hydrodynamic equations via multi-domain spectral methods
- We compare different numerical methods by comparing the accuracy with which they compute GRV2
- Outlook
  - The code to compute GRV2 in AKM is almost, but not yet complete. Once this is done we can compare AKM to LORENE to determine which initial data solver is more accurate.
  - Eventually we will begin time-evolution of neutron star binaries modeled with AKM/LORENE.

# References I

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