

# Revisiting ‘Sidebands of Sidebands’ in FINESSE

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# 1 Introduction

This document describes an upgrade of the interferometer simulation tool FINESSE [1]. With the release of FINESSE version 1.1 on 04/11/2013, the modelling of so-called ‘sidebands of sidebands’ has been improved. To simulate these objects in previous versions of FINESSE, one had to use a series of workarounds, as documented in a note by Keita Kawabe [4]. Although the note still provides a useful introduction to the topic, the workarounds described there are no longer necessary.

## 1.1 The Sideband Picture

Laser interferometers for precision length measurements, such as interferometer gravitational wave detectors, can offer a rich and complex field for study. Many such investigations can be performed assuming a steady state of the interferometer and only linear optical couplings, such that the light field can be described in the frequency domain. In the following, we assume a stable central laser frequency, and describe all other frequency components as sidebands (i.e. components with an offset to the laser frequency).

Such sidebands are generated, for example, via the modulation of the laser field, that is, with a periodic change of a parameter of the light field, such as phase or amplitude [3]. We distinguish ‘modulation sidebands’, created by electro-optical modulators (EOMs) for control purposes (typically at radio frequency), and ‘signal sidebands’ (also sometimes called ‘audio-sidebands’), which are generated by small changes in the optical set-up, for example, the oscillation of a mirror position. When a light field containing sidebands is modulated again, we say that we have created ‘sidebands of sidebands’ (SoS).

## 1.2 Sidebands in FINESSE

Earlier versions of FINESSE (1.0, and any before that) computed the various frequency components of the light fields sequentially. First the carrier frequencies were considered, then the sidebands from EOMs, and finally the signal sidebands (created with the `fsig` command). The purpose of this separation was a decrease in the computational complexity and thus faster simulation times. Unfortunately, it limited FINESSE such that it could not compute the full couplings between different types of frequency components. Sidebands of sidebands were generated in a limited fashion. While the majority of simulation tasks do not require a more complete treatment, certain cases were difficult or impossible to model [4].

As of FINESSE 1.1, all components are computed simultaneously and all couplings are described correctly. Hence, SoS are now implemented correctly. It is, however, still necessary to limit the number of frequency components; otherwise for example, a modulator in a cavity would create an infinite number of components. The user can ‘turn on/off’ certain SoS so that FINESSE retains the balance between speed and flexibility.

In time with the recent upgrade, the purpose of this note is to highlight the new implementation of sideband coupling in FINESSE 1.1. With a few examples, we illustrate to what extent SoS may be added (or removed) from a simulation and how this can affect the output. Whilst on the subject, we also take the opportunity to demonstrate how the `fsig` command can now be used with a laser directly, in order to generate signal sidebands.

## 2 Producing Sidebands of Sidebands

One simple method of generating SoS is by the set-up in Figure 1. Here, two sequential EOMs apply phase modulations to a laser beam. Although this set-up would only generate modulation sidebands, the following mathematical analysis applies equally to signal sidebands.

If one assumes that the laser produces a monochromatic field

$$E_{\text{in}} = E_0 e^{i\omega_0 t}, \quad (1)$$

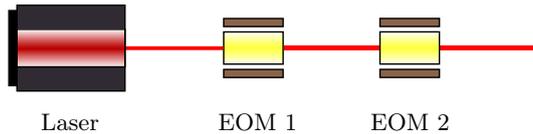


Figure 1: A simple set-up for generating SoS. Firstly, a field  $E_{\text{in}}$  passes through EOM 1. For each frequency component in this field, (modulation) sidebands are generated with offsets of  $\pm p\Omega_1$ , where  $p$  is a positive integer. On passing through EOM 2, each component then gains its *own* (modulation) sidebands, each with frequency offsets of  $\pm q\Omega_2$ , where  $q$  is a positive integer. With respect to the carrier, the (modulation) SoS are offset by angular frequencies  $\pm p\Omega_1 \pm q\Omega_2$ .

then after the EOMs, the field will, up to a phase factor, be given by

$$E_{\text{out}} = E_0 e^{i\omega_0 t} e^{im_1 \cos(\Omega_1 t + \varphi_1)} e^{im_2 \cos(\Omega_2 t + \varphi_2)}. \quad (2)$$

Here,  $m_i$  are the modulation indices,  $\Omega_i$  are the (phase) modulation frequencies,  $\varphi_i$  are the modulation phases and  $i = \{1, 2\}$  denotes which EOM these quantities are associated with. Expanding this to first-order in  $m_1$ ,  $m_2$  and the product  $m_1 m_2$ , and writing cosines as exponentials, one finds that

$$E_{\text{out}} = \sum_{n=0}^N a_n e^{i\omega_n t} = E_0 \left[ \underbrace{e^{i\omega_0 t}}_{\text{carrier}} + \frac{im_1}{2} \left( \underbrace{e^{i(\omega_0 + \Omega_1)t}}_{\text{first set of sidebands}} + \underbrace{e^{i(\omega_0 - \Omega_1)t}}_{\text{first set of sidebands}} \right) + \frac{im_2}{2} \left( \underbrace{e^{i(\omega_0 + \Omega_2)t}}_{\text{second set of sidebands}} + \underbrace{e^{i(\omega_0 - \Omega_2)t}}_{\text{second set of sidebands}} \right) - \frac{m_1 m_2}{4} \left( \underbrace{e^{i(\omega_0 + \Omega_1 + \Omega_2)t}}_{\text{sidebands of sidebands}} + \underbrace{e^{i(\omega_0 + \Omega_1 - \Omega_2)t}}_{\text{sidebands of sidebands}} + \underbrace{e^{i(\omega_0 - \Omega_1 + \Omega_2)t}}_{\text{sidebands of sidebands}} + \underbrace{e^{i(\omega_0 - \Omega_1 - \Omega_2)t}}_{\text{sidebands of sidebands}} \right) \right]. \quad (3)$$

Written in this way, one can clearly see the carrier, the first set of sidebands, the second set of sidebands, and the sidebands of sidebands, as well as their respective amplitudes. For additional clarity, a pictorial representation of this field is shown in Figure 2.

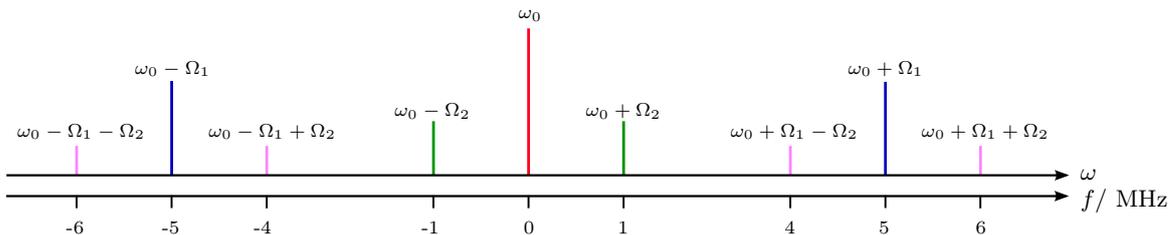


Figure 2: The ‘line spectrum’ of the field  $E_{\text{out}}$  for  $\Omega_1 > \Omega_2$ . Labels above the lines give the corresponding analytical angular frequencies, as seen in equation (2). Labels below each line denote their numerical frequency offset from the carrier frequency, and are given in MHz. This latter set of values are those handled by FINESSE in the first of our examples discussed below.

### 3 Detecting Sidebands of Sidebands

To illustrate the presence of SoS in FINESSE, we simulate photodetection via demodulation of the field  $E_{\text{out}}$ . This scheme allows us to see the contribution to the power incident on a photodiode from the components of the field, as specified in terms of their frequency<sup>1</sup>. Following [3], the intensity of a field with components of frequencies  $\omega_n$  may be written as

$$S_{\text{out}} = \frac{c\epsilon_0}{2} |E_{\text{out}}|^2 = \frac{c\epsilon_0}{2} \sum_{n=0}^N \sum_{m=0}^N a_n a_m^* e^{i(\omega_n - \omega_m)t}. \quad (4)$$

<sup>1</sup>Note that we refer to *intensities* in the text, and yet FINESSE handles *power*. These are numerically the same, and equivalent for our purposes. A technical discussion of this detail may be found the FINESSE manual, found at <http://www.gwoptics.org/finesse>.

Passing this through mixer with we obtain a signal proportional to

$$S'_{\text{out}} = S_{\text{out}} \cdot \cos(\omega_x t + \varphi_x), \quad (5)$$

where  $\omega_x$  is the demodulation frequency and  $\varphi_x$  is the demodulation phase and the DC part is given by

$$S'_{\text{out,DC}} = \frac{c\epsilon_0}{2} \sum_{n=0}^N \sum_{m=0}^N \text{Re} (a_n a_m^* e^{-i\varphi_x}) \quad \text{with } \omega_x = \omega_n - \omega_m. \quad (6)$$

This is the quantity that we measure (by simulation) in the first two examples. In the third example, an additional demodulation is performed, the details of which are saved until then.

## 4 A Simple Example

In the following example, SoS are generated with the set-up shown in Figure 1. The EOMs are set such that  $\Omega_1 = 2\pi \cdot 5$  MHz and  $\Omega_2 = 2\pi \cdot 1$  MHz, Figure 2 may be used as a reference.

Demodulation is performed with  $\omega_x = 2\pi \cdot 6$  MHz and  $\varphi_x = 0$ . As one can see from Figure 2, a difference of  $f = 6$  MHz appears four times. The first two are between the carrier and the  $\pm 6$  MHz SoS. The second two are between the first and second sidebands of the carrier, that is, between  $\pm 1$  MHz and  $\mp 5$  MHz. The contributions of these two pairs exactly cancels and the output signal is zero:

$$S'_{\text{out,DC}} = \frac{c\epsilon_0}{2} [\text{Re}(a_0 a_8^*) + \text{Re}(a_5 a_0^*) + \text{Re}(a_3 a_2^*) + \text{Re}(a_1 a_4^*)] = 0. \quad (7)$$

This result is simulated by the FINESSE code given below, and the output is shown by the blue trace in Figure 3. Here, we arbitrarily tune  $\varphi_1$  the modulation phase of EOM 1 in order to produce a 2D plot.

```
# FINESSE 1.1                               Build 1.1-21-ga21fe41
mf -6M -4M 4M 6M                             # Instructing Finesse to add SoS
l L1 1 0 n0                                  # Laser: 1W
mod EOM1 5M 0.05 1 pm 0 n0 n1              # EOM1: Omega_1 = 2pi*5 MHz, m_1 = 0.05
s S1 0 n1 n2                                 # Zero space
mod EOM2 1M 0.01 1 pm 0 n2 n3             # EOM2: Omega_2 = 2pi*1 MHz, m_2 = 0.01
pd1 PD 6M 0 n3                              # PD: omega_x = 2pi*6 MHz
xaxis EOM1 phase lin 0 360 100             # Tuning the modulation phase of EOM1
yaxis abs                                    # Plotting the amplitude
```

The first line of code requests the creation of modulation SoS. One simply specifies their frequency offsets from the carrier and FINESSE does the rest of the work. If these modulation SoS are not of interest, then one may simply switch them off by commenting out this first line. In this case, the first two terms in equation (7) do not appear, and the result is non-zero. The output of this simulation is shown by the red trace in Figure 3.

Clearly, there is a difference in output depending on whether these SoS are simulated or not, and only the result including SoS is correct. However, this is due to the demodulation frequency being specifically chosen to illustrate this effect. In many practical cases SoS can be omitted (to increase the speed of simulations) without any change in the detector outputs.

## 5 Sidebands of Sidebands in a Pound-Drever-Hall Set-up

In the simple example above, it was shown how SoS can now be modelled properly in FINESSE. In this section we turn to a more practical application: SoS in a Pound-Drever-Hall (PDH) set-up [2]. We show how SoS can complicate an error signal, and how a transfer function of laser noise into a PDH signal is now more easily produced in FINESSE.

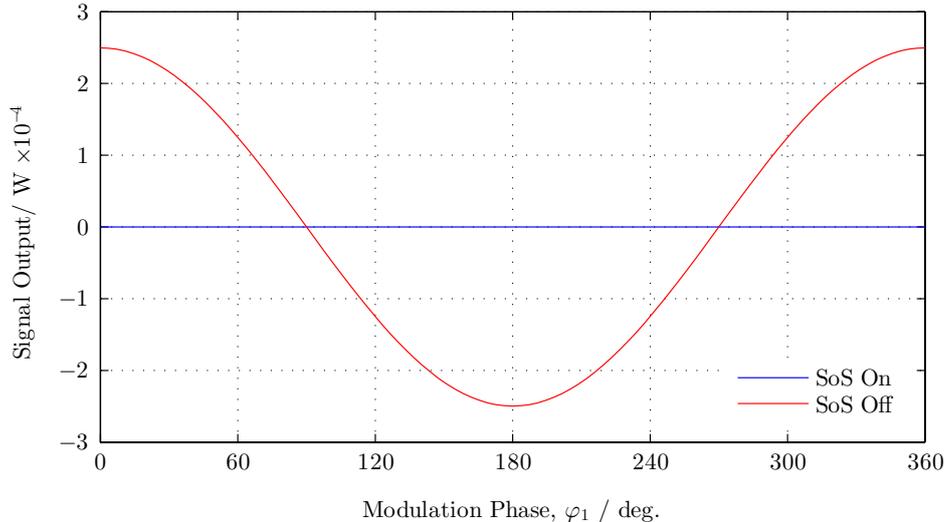


Figure 3: Photodiode output as a function of modulation phase at EOM 1. This is a simple demonstration of the difference in result when SoS are ‘turned on/off’ in FINESSE. In this example (see section 4), the contribution to the photodiode signal from terms due to SoS cancel those terms that do not originate from SoS, and the result is a zero output, as shown by the blue trace. When SoS are not simulated (when the line `mf -6M -4M 4M 6M` is removed), this cancellation is not modelled, and an incorrect, non-zero result is obtained, as shown by the red trace.

## The Error Signal

Our error signal is generated by the set-up shown in Figure 4. Here, SoS are again produced by a pair of EOMs.

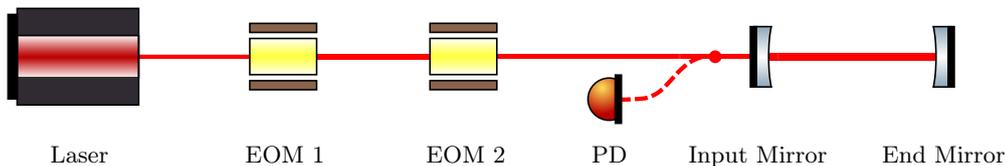


Figure 4: A schematic representation of a set-up used to generate a PDH error signal. Note that this is drawn in such a way as to match what FINESSE sees. As the carrier deviates from resonance with the cavity, the photodiode measures a changing intensity. Combined with the sidebands whose offsets are  $\pm\Omega_2$ , the photodiode measures a beat pattern. The side of the resonance that the carrier is on is encapsulated its phase between these sidebands. The first EOM, and hence the SoS, are ‘added complications’ that one may have in a real interferometer.

Suppose that the field incident on the cavity has components of angular frequency  $\omega_n$  with respective amplitudes  $a_n$ . The reflected field is then

$$E_{\text{out}}(\omega) = \sum_{n=0}^N F(\omega_n) a_n e^{i\omega_n t}. \quad (8)$$

Here, the weighting factor is the reflection coefficient of the cavity. In this document, we consider a lossless cavity, for which

$$F(\omega) = \frac{-\sqrt{R_1} + \sqrt{R_2} e^{i\omega/\Delta\nu_{\text{FSR}}}}{1 - \sqrt{R_1 R_2} e^{i\omega/\Delta\nu_{\text{FSR}}}}, \quad (9)$$

where  $R_i$  and  $T_i$  are the reflection and transmission coefficients of mirror  $i$ , such that  $R_i + T_i = 1$ . In this particular example, we consider a symmetric cavity. In this case, when the cavity is mode-matched to a component of the field with angular frequency  $\omega$ , then  $F(\omega) = 0$  and this component is absent from  $E_{\text{out}}$ . This

condition is given by

$$\frac{\omega}{\Delta\nu_{\text{fsr}}} + 2\phi = 2\pi N. \quad (10)$$

Here,  $\Delta\nu_{\text{fsr}}$  is the free spectral range of the cavity,  $\phi$  is a tuning parameter of the end mirror, and  $N$  is the set of integers.

The power on the photodiode is calculated from equation (8), and in the same way as above, demodulation is performed at angular frequency  $\omega_x$ . Denoted by  $\varepsilon$ , we then take the PDH error signal as the DC part of this result. In this example, the error signal is generated by tuning the quantity  $\phi$  from  $-15^\circ$  to  $+15^\circ$ , as seen in the corresponding simulation code below.

```
# FINESSE 1.1                               Build 1.1-21-ga21fe41
mf -11M -3M 3M 11M                          # Instructing Finesse to add SoS
l L1 1 0 n0                                  # Laser: 1W
mod EOM1 7M 0.6 1 pm n0 n1                 # EOM1: Omega_1 = 2pi*7 MHz, m_1 = 0.6
s S1 0 n1 n2                                # Zero space
mod EOM2 4M 0.6 1 pm n2 n3                 # EOM2: Omega_2 = 2pi*4 MHz, m_2 = 0.6
s S2 0 n3 n4                                # Zero space
m M1 0.99 0.01 0 n4 n5                     # Mirror1: R = 0.99, T = 0.01, phi = 0
s S3 1200 n5 n6                             # Space: L = 1200 m
m M2 0.99 0.01 0 n6 dump                   # Mirror2: R = 0.99, T = 0.01, phi = 0
pd1 inphase 4M 0 n4                         # PD: omega_x = 2pi*4 MHz
xaxis M2 phi lin -15 15 400                # Tuning the cavity length through phi
yaxis abs                                    # Plotting the amplitude
```

Demodulation is performed with  $\omega_x = \Omega_2 = 2\pi \cdot 4$  MHz and  $\varphi_x = 0$ . Making the identity  $F(\omega_n) \equiv F_n$ , and working to first order in  $m_1$ ,  $m_2$ , and the product  $m_1 m_2$ , the error signal is found to be

$$\varepsilon = -\frac{c\varepsilon_0 E_0^2 m_2}{16} \left[ 4 \operatorname{Im}(F_2 F_0^* + F_4 F_0^*) + m_1^2 \operatorname{Im}(F_5 F_1^* + F_6 F_1^* + F_7 F_2^* + F_8 F_2^*) \right]. \quad (11)$$

Note that this quantity contains a contribution from every component of the field and that terms with SoS in them are prefixed by a small number. The output is shown in Figure 5, with the blue trace for when SoS are simulated and the red trace for when they are not simulated.

When a component of the field meets the resonance condition, a kink is produced in the error signal. The value of  $\phi$  for a particular component is calculable directly from equation (10). The resonances of the SoS add another layer of complexity to the error signal. For the purposes of locking, however, we see that they are not interesting and do not need to be simulated.

Recall that, in reality, an EOM produces sidebands with angular frequency offsets  $\pm p\Omega$ , i.e. a whole series of harmonics. The effect on a photodiode of higher harmonics will in general be greater than that from SoS, and this can also be simulated with FINESSE. Here, however, we simulate only the first harmonics. While the output is therefore erroneous, the purpose is merely to illustrate the effect of SoS in an independent way.

## The Transfer Function

With the set-up in Figure 6, we now turn to the analysis of frequency noise by deriving of the transfer function of a simple cavity. Not only is this a common simulation task for FINESSE, it is also one where earlier versions required complicated workarounds in order to achieve (e.g. in adding laser noise by shaking a dummy beam splitter after, but not before, the EOM).

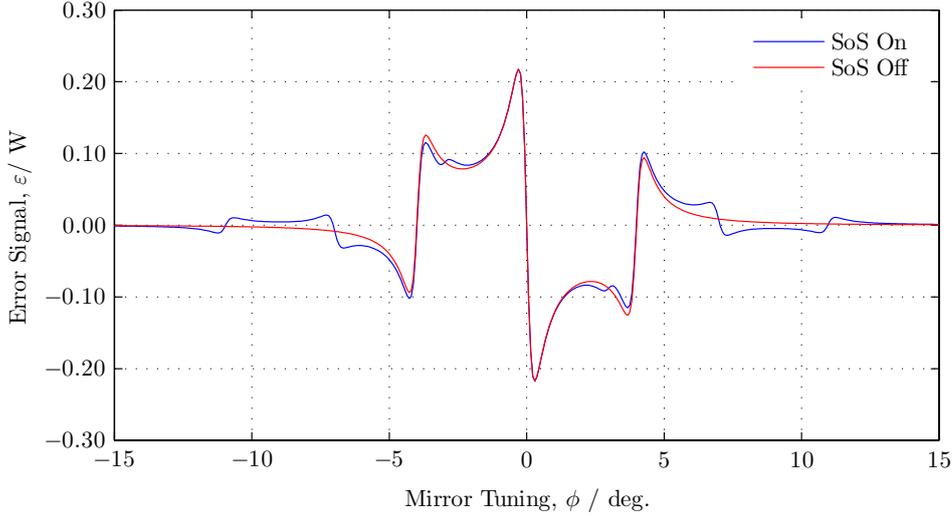


Figure 5: PDH error signals generated by the above FINESSE simulation. The error signal provides a measure of the intensity reflected from the cavity, as a function of the carrier’s perturbation from resonance. Its asymmetry derives from the phase between the carrier and the sidebands at the demodulation frequency, and allows us to determine whether the carrier is above or below resonance. The red trace shows the simulations where the second EOM does not generate SoS. In the blue trace, these SoS are generated, and appear as kinks in the error signal when they resonate.

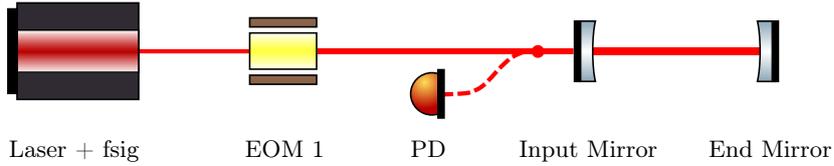


Figure 6: A schematic representation of a set-up used to generate a transfer function for frequency noise reflected from a cavity. Note that this is drawn in such a way as to match what FINESSE sees. To generate signal sidebands at  $\Omega_1$ , **fsig** is applied *directly* to the laser. The EOM then creates modulation sidebands at  $\pm\Omega_2$  around the carrier and also around the signal sidebands (giving SoS). In this case, we have  $\Omega_1 \ll \Omega_2$ . The photodetection is extended by an additional mixer to demodulate at the signal frequency. The transfer function is obtained as we sweep over this frequency, simultaneously at the laser and the second mixer [3].

The aim is to measure the effect of laser frequency or laser phase noise on a PDH error signal. The **fsig** command is used to apply a sinusoidal phase modulation with parameters  $m_1$  and  $\Omega_1$ . In this case, the incoming field is approximated by equation (3), and the reflected field by equation (8). In this example we have *two* local oscillators, and so

$$S'_{\text{out}} = \frac{1}{2} S_{\text{out}} \cos[(\omega_y + \omega_x)t + (\varphi_y + \varphi_x)] + \frac{1}{2} S_{\text{out}} \cos[(\omega_y - \omega_x)t + (\varphi_y - \varphi_x)], \quad (12)$$

We set  $\varphi_x = 0$  but do not specify  $\varphi_y$ . In this case, the output is a complex number given by

$$S'_{\text{out,complex}} = \frac{c\varepsilon_0}{2} \left( \sum_{n=0}^N \sum_{m=0}^N b_n b_m^* + \sum_{p=0}^N \sum_{q=0}^N b_p b_q^* \right) \quad \text{with} \quad \begin{cases} \omega_y + \omega_x = \omega_n - \omega_m \\ \omega_y - \omega_x = \omega_p - \omega_q. \end{cases} \quad (13)$$

Here,  $b_i = F_i a_i$  and the frequencies are  $\omega_x = \Omega_2$  and  $\omega_y = \Omega_1$ . From this, we obtain

$$S'_{\text{out,complex}} = \frac{c\varepsilon_0 E_0^2 m_1 m_2}{8} (F_1 F_4^* + F_3 F_2^* - F_0 F_8^* - F_5 F_0^* + F_4 F_2^* + F_1 F_3^* - F_6 F_0^* - F_0 F_7^*), \quad (14)$$

where  $m_1 = 1 \text{ Hz}/\Omega_1$  and where  $F(\omega)$  is given by equation (9). Our transfer function, and final result, is then

$$H(\omega) = 20 \log_{10} \left| \frac{S'_{\text{out,complex}}}{1 \text{ Wm}^{-2}} \right|. \quad (15)$$

The corresponding simulation code is provided below.

```
# FINESSE 1.1                                Build 1.1-21-ga21fe41

l L1 1 0 n0                                    # Laser: 1W
fsig F1 L1 10 0                               # Adding frequency noise to the laser
s S1 0 n0 n1                                   # Zero space
mod EOM 10M 0.01 1 pm 0 n1 n2                # EOM: Omega_2 = 2pi*10 MHz, m_2 = 0.01
s S2 0 n2 n3                                   # Zero space
m M1 0.99 0.01 0 n3 n4                       # Mirror1: R = 0.99, T = 0.01, phi = 0
s S3 1200 n4 n5                               # Space: 1200 m
m M2 0.9999 0.0001 0 n5 dump                 # Mirror2: R = 0.9999, T = 0.0001, phi = 0

pd2 PD 10M 0 10 n3                           # Double demodulation

xaxis F1 f log 1 1000 4000                   # Tuning the frequency noise
put PD f2 $x1                                 # Tuning omega_y correspondingly
yaxis db                                      # Plotting the amplitude in dB
```

The sidebands created by the `fsig` command are signal sidebands. As of FINESSE 1.1, EOMs now automatically couple signal sidebands with modulation sidebands. So, in addition to creating modulation sidebands around the carrier, new signal sidebands are created around the modulation sidebands. In this simulation, the interesting comparison is not between SoS ‘on’ and SoS ‘off’. Rather, we note the simplicity of producing a transfer function, as compared with versions of FINESSE before 1.1. That `fsig` can now be applied directly on to a laser not only makes for a more intuitive simulation, but also reduces its complexity. The FINESSE output is given by the blue trace in Figure 7, with the dashed red trace showing the theoretical prediction from equation (15).

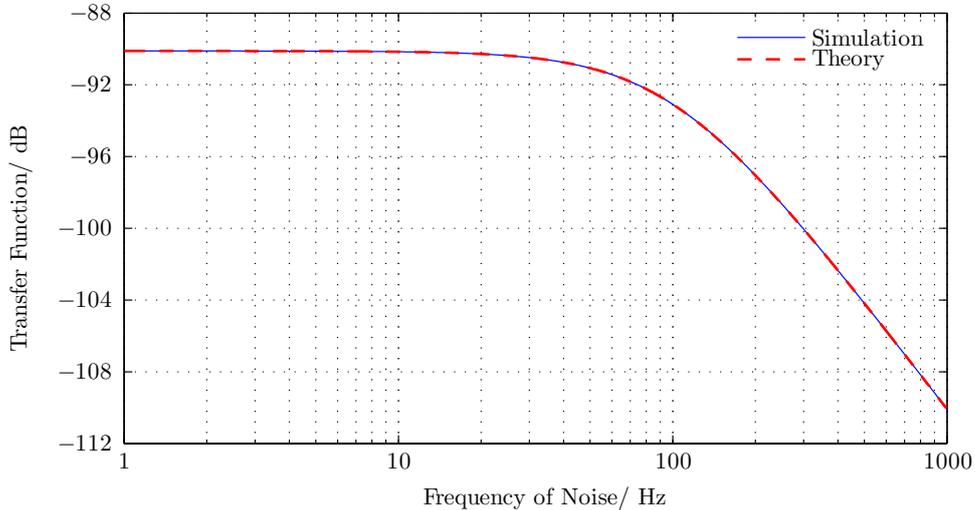


Figure 7: The transfer transfer function for the detection of frequency noise reflected from a cavity. The dashed red trace shows the theoretical prediction from equation (15), and the blue trace shows the output of FINESSE, from the code given above. The main point of interest is not the overlay, but rather the simplicity under which this result may be produced by FINESSE 1.1.

## References

- [1] Freise, A.; Heinzl, G.; Lück, H.; Schilling, R.; Willke, B. and Danzmann, K., ‘Frequency-domain interferometer simulation with higher-order spatial modes’, *Classical and Quantum Gravity*, **21**, S1067-S1074 (2004). The software is available at <http://www.gwoptics.org/finesse>
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