Higher-order gravitational wave emission in core-collapse supernovae

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Abstract. Gravitational waves are important messengers that carry information about the multi-dimensional fluid dynamics in the central engines of core-collapse supernovae. So far, most simulations use the so-called quadrupole formalism to extract the waves from the matter dynamics, but recent results suggest that significant emission may also occur at higher than quadrupole order. Comparing the relative sizes of the first- and second-order terms would yield a quantitative measure of how well the quadrupole moment represents the gravitational waves. In this report, I derive the octupole order terms and express them in a manner suitable for simulation. These correction terms were added to the simulations, and initial testing has been performed on an excited Tolman-Oppenheimer-Volkoff star. We discuss future work needed for conclusive assessment of their significance.

1. Introduction

The physics community is very interested in observing gravitational waves because we hope that these waves will give us a new way of investigating astrophysical bodies. Different parts of the electromagnetic spectrum give us different information about the universe. Each time we have opened up an additional portion of that spectrum for observation, we have gained new insight into astrophysics. Gravitational radiation is a previously unexplored spectrum that astronomers can use to probe the cosmos. For example, gravitational waves are expected to yield information about the internal dynamics of sources (like supernovae) that light may not communicate. Another advantage of this new spectrum over light is that gravitational waves interact very weakly with the matter between the source and the detector meaning that little information is lost in transit. In addition to increasing our understanding of celestial bodies, gravitational waves provide tests of general relativity. Einstein's theory predicts the existence and behavior of gravitational waves and thus their detection would test his theory. Although there are numerous candidate sources for gravitational waves, this project is concerned with the waves produced by supernovae.

Sometimes when stars reach the end of their lives they explode. If the explosion is particularly bright, it is called a supernova. The exact mechanism of these explosions is not fully understood. Since physicists cannot create supernovae in laboratories, they create computer simulations of them to examine their behavior. Supernovae are especially interesting because of they incorporate a wide variety of physical processes including nuclear, electromagnetic, and gravitational interactions. For example, nuclear physics plays an important role in the neutron stars created by supernovae; the evolution of supernovae is greatly influenced by internal electromagnetic forces; and supernovae undergo strong gravitational interactions and are predicted to produce gravitational waves. Computer simulations indicate that when a supernova occurs it provides a brief source of gravitational waves strong enough that they may be detected by distant gravitational wave observatories. Observations of the gravitational radiation produced by supernovae may give new understanding of exploding stars, although there have never been any direct observations yet.

Current simulations calculate the gravitational wave emissions from supernovae using the quadrupole formalism [1]. The quadrupole formalism consists of taking a multipole expansion of the object observers use to measure distance and time (called the metric perturbation). This expansion is similar to the Taylor series $f(x+\varepsilon) = f(x) + \varepsilon f'(x) + \frac{1}{2}\varepsilon^2 f''(x) + ...$ where the function f is analogous to the metric perturbation and ε is a small parameter that, in our case, is analogous to the length of the source divided by the distance between the source and the observer. In the quadrupole approximation, the gravitational waves are calculated based on the first term in the multipole expansion. Since the quadrupole moment is the first term in the multipole expansion, the quadrupole moment is called a first-order approximation for the gravitational waves.

The purpose of this SURF is to investigate the contribution to the gravitational waves from the second-order multipole moments. These moments are called the mass octupole and current quadrupole moments. These second-order corrections are of interest because they provide a quantitative measure of how well the quadrupole moment represents the gravitational radiation. Small correction terms indicate that the gravitational waves are dominated by the contributions from the quadrupole moment. However, if the correction terms are large compared to the first order approximations, then the application of the quadrupole formalism to supernovae may need to be reevaluated.

This report is organized as follows. In Section 2, I describe how to extract the quadrupole and octupole order gravitational wave signal from a matter source. In Section 3, the multipole moments obtained in the previous section are cast in a form appropriate for simulation. Section 4 describes the implementation of the octupole order moment in a test simulation.

2. Gravitational waves beyond the linear approximation

The purpose of this section is to review the relaxed Einstein equations and obtain the multipole moments of the gravitational radiation without assuming a flat spacetime background. We begin by defining the metric perturbation $h^{\mu\nu}$ as

$$h^{\mu\nu} = \eta^{\mu\nu} - \sqrt{-g}g^{\mu\nu} \ , \tag{1}$$

where g is the determinant of $g_{\mu\nu}$. Unlike in linearized gravity where the background spacetime is flat, we do not require that $|h_{\mu\nu}| \ll 1$. Gravity plays an important role in the physics governing stellar evolution, in particular supernovae are self-gravitating sources with strong internal gravity compared to the gravity in the waves. For this reason, we do not require that a flat background spacetime or that $|h_{\mu\nu}| \ll 1$. The Einstein equations are equivalent to the following two conditions called the relaxed Einstein equations [2]

$$\partial_\mu h^{\mu\nu} = 0 \ , \tag{2}$$

$$\Box h^{\mu\nu} = -16\pi[(-g)T^{\mu\nu} + t^{\mu\nu}] \ . \tag{3}$$

Here $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the flat spacetime d'Alembertian, $T^{\mu\nu}$ is the energy-momentum tensor from Einstein's equation, and

$$t^{\mu\nu} = (-g)t^{\mu\nu}_{LL} + \frac{1}{16\pi}\left[(\partial_\beta h^{\mu\alpha})(\partial_\alpha h^{\nu\beta}) + (\partial_\alpha\partial_\beta h^{\mu\nu})h^{\alpha\beta}\right] \ , \tag{4}$$

where $t^{\mu\nu}_{LL}$ is the Landau-Lifschitz pseudotensor. (For a definition and description of $t^{\mu\nu}_{LL}$ see [2] page 118 and [3] page 466.) The relaxed Einstein equations (2) and (3) are mathematically equivalent to Einstein's equation; we rearranged the terms in Einstein's equation but no approximations were made. For convenience we define the effective energy-momentum tensor $T^{\mu\nu}_{eff} = (-g)T^{\mu\nu} + t^{\mu\nu}$. Differentiating (3) with respect to x^μ and applying $\partial_\mu h^{\mu\nu} = 0$, we see that $T^{\mu\nu}_{eff}$ is conserved

$$\partial_\mu T^{\mu\nu}_{eff} = \partial_\mu[(-g)T^{\mu\nu} + t^{\mu\nu}] = 0 \ . \tag{5}$$

This conservation equation will be useful later when simplifying expressions for the multipole moments.

It will also be useful to know the relative sizes of the components of $T^{\mu\nu}_{eff}$ (see [3] Exercise 20.5). The energy-momentum tensor from Einstein's equation, $T^{\mu\nu}$, represents the matter distribution while $t^{\mu\nu}$ represents the energy-momentum from the gravitational field. Since gravitational fields are massless, their mass-like components $t^{\mu 0}$ should be much smaller than the mass-like components from $T^{\mu 0}$. Hence, we will take $t^{\mu 0} \ll T^{\mu 0}$. We will take T^{ij} and t^{ij} to be of the same order of magnitude, for which we provide the following heuristic argument [4].

Consider a spinning dumbbell, that is, two large weights connected by a rod in the middle. Furthermore, let $t^{\mu\nu}$ be the energy-momentum tensor of the rod and let $T^{\mu\nu}$ be the energy-momentum of the weights. The mass of the rod is much less than the mass of the weights so $t^{00} \ll T^{00}$ as we argued above. The momentum of the rod is very small compared to the momentum of the weights again because the rod is far less massive than the weights. In other words, $t^{0j} \ll T^{0j}$ as we argued in the previous paragraph. (Note that the total momentum of the dumbbell would be zero, but the inequality holds for any small piece of the dumbbell.) Finally, we claim that the pressure and shear components of the rod and weights are roughly comparable, that is, we claim $t^{ij} \sim T^{ij}$. Consider the point where the rod contacts the weight. The reason the dumbbell doesn't break is that the pressure from the rod on the weight at the contact point must balance the pressure from the weight on the rod, i.e. t^{ij} and T^{ij} are of the same order of magnitude. The exact motion of the dumbbell is irrelevant to the argument as is the shape of the object. The important feature is that we have a set of heavy objects (the weights in the example above) connected or supported by a set of light objects (the rods). This analogy carries over to the stars we wish to simulate where the matter of the star plays the role of the weights and gravity plays the part of the rods holding the matter together. There is a small caveat in applying this argument to a supernova, namely that the star is exploding. However, we will assume that the motion is not too rapid so that t^{ij} and T^{ij} are still comparable in size. Since the evolution equation (3) for t^{ij} is non-linear, we want to avoid calculating it. The point of these heuristic arguments is that if we can re-express $h^{\mu\nu}$ in terms of $T^{0\mu}_{eff}$ then we can take $T^{0\mu}_{eff} = (-g)T^{0\mu}$ and avoid calculating $t^{\mu\nu}$. In this sense, we use $t^{00} \ll T^{00}$ and $t^{0j} \ll T^{0j}$ but we never explicitly use how t^{ij} and T^{ij} are related in size except that we suspect that we cannot ignore t^{ij} in favor of T^{ij}.

We can solve the wave equation (3) for $h^{\mu\nu}$ using a Green's function. The solution is the non-linear integral equation

$$h^{\mu\nu}(t, \mathbf{x}) = 4 \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T^{\mu\nu}_{eff}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}') \ , \tag{6}$$

where $\mathbf{x}$ is the position of the observer. The radiative parts of $h^{\mu\nu}$ can be obtained by taking the transverse-traceless (TT) projection of $h^{\mu\nu}$. Only the spatial components h^{ij} of $h^{\mu\nu}$ contribute to radiation so we only need to consider the case where $(\mu, \nu) = (i, j)$. We obtain the multipole moments by expanding $|\mathbf{x} - \mathbf{x}'|$ and $|\mathbf{x} - \mathbf{x}'|^{-1}$ in power series in terms of $x'/x \ll 1$ and taking a Taylor series of $T^{\mu\nu}_{eff}$. After performing these expansions (see the Appendix), the two largest contributions to $h^{TT}_{\mu\nu}$ are

$$h^{ij}_{TT} = \frac{4}{x}[S^{ij} + n_k \partial_t S^{ijk}]^{TT}_{t-x} \ , \tag{7}$$

where $\mathbf{n} = \hat{\mathbf{x}}$, the subscript $t - x$ means that the energy-momentum tensors need to be evaluated at the retarded time $t - x$, and the tensors S^{ij} and S^{ijk} are defined as

$$S^{ij} = \int d^3x' \ T^{ij}_{eff} \ , \qquad \text{and} \qquad S^{ijk} = \int d^3x' \ x'^k T^{ij}_{eff} \ . \tag{8}$$

The term S^{ij} represents the mass quadrupole moment while S^{ijk} represents the combination of the mass octupole and current quadrupole moments. We will also refer to S^{ijk} as the second order moment and the octupole moment. Computing S^{ij} and S^{ijk} as they are written above involves calculating the t^{ij} components of T^{ij}_{eff}. We prefer to avoid calculating t^{ij} because the expression for t^{ij} is complicated but more importantly the inclusion of t^{ij} makes (6) a non-linear equation.

3. Multipole moments suitable for simulation

In this section, we derive expressions for S^{ij} and S^{ijk} that do not involve t^{ij}. Since these expressions make no reference to t^{ij}, they are appropriate to include in computer simulations. The general method for eliminating t^{ij} from the multipole moments is to apply energy-momentum conservation $\partial_\mu T^{\mu\nu}_{eff} = 0$ and then integrate by parts.

Before re-expressing S^{ij} we should rewrite the conservation of energy-momentum. Applying $t^{\mu 0} \ll T^{\mu 0}$ to the conservation of energy-momentum equation $0 = \partial_\mu T^{\mu j} = \partial_n T^{nj} + \partial_t T^{0j}$ we obtain

$$\partial_n T^{nj}_{eff} + \partial_t(-gT^{0j}) = 0 \ . \tag{9}$$

Using the previous equation, we can convert S^{ij} into a more useful form as follows:

$$\begin{aligned} S^{ij} &= \int d^3x' \ T^{ij}_{eff} \\ &= \int d^3x' \ T^{ij}_{eff} + \int d^3x' \ x'^i \left[\partial_n T^{nj}_{eff} + \partial_t(-gT^{0j})\right] \\ &= \int d^3x' \ \left[T^{ij}_{eff} + x'^i \partial_n T^{nj}_{eff}\right] + \partial_t \int d^3x' \ (-g)T^{0j}x'^i \\ &= \int d^3x' \ \partial_n(x'^i T^{nj}_{eff}) + \partial_t \int d^3x' \ (-g)T^{0j}x'^i \ . \end{aligned} \tag{10}$$

In the last line, the integral over $\partial_n(x'^i T^{nj}_{eff})$ can be converted to a surface integral over $x'^i T^{nj}_{eff} = x'^i(-gT^{nj} + t^{nj})$. We will now show that the surface integral vanishes. Since the source is isolated, T^{ij} is zero outside the source so it does not contribute to the surface integral. However, the surface integral of $x'^i t^{nj}$ contains terms involving h^{nj} and x'^i, both of which are generally non-zero outside the source. Far from the source, however, the metric is nearly given by the Newtonian line element $ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)\delta_{ij}dx^i dx^j$ so that $h^{\mu\nu}$ is of order $\Phi\delta^{\mu\nu}$. If the mass of the source is M and the distance to the source is R then $h^{\mu\nu} \sim \Phi \sim M/R$. According to (4), we know that t^{ij} is of order $(\partial_\alpha h^{\mu\nu})(\partial_\beta h^{\sigma\rho}) \sim M^2/R^4$. Finally the surface integral of $x'^i t^{nj}$ contains a factor of R^2 from the surface area, a factor of R from x'^i, and a factor of M^2/R^4 from t^{nj} so the surface integral goes like M^2/R which vanishes for large radii. Another way to conclude that the surface integral vanishes is to put the surface of integration so far from the source that disturbances to the background metric due to the presence of the star will never reasonably reach the integration surface in the life of the star or in the time it takes the gravitational waves to reach the observer. For example, putting the integration surface at a radius equal to the distance light travels in the age of the universe should guarantee that the surface integral is zero. Whatever justification we choose to use, we will assume that the surface integral in (10) goes to zero leaving us with

$$S^{ij} = \partial_t \int d^3x' \ (-g) T^{0j} \ . \tag{11}$$

To reformulate the mass octupole and current quadrupole moment we perform a similar procedure. Consider

$$\begin{aligned} S^{ijk} &= \int d^3x' \ T^{ij}_{eff} x'^k \\ &= \frac{1}{2} \int d^3x' \ (T^{ij}_{eff} x'^k + T^{ik}_{eff} x'^j + T^{ij}_{eff} x'^k + T^{jk}_{eff} x'^i - T^{ik}_{eff} x'^j - T^{jk}_{eff} x'^i) \\ &= \frac{1}{2} \int d^3x' \ (T^{mi}_{eff} \partial_m (x'^j x'^k) + T^{mj}_{eff} \partial_m (x'^i x'^k) - T^{mk}_{eff} \partial_m (x'^i x'^j)) \\ &= -\frac{1}{2} \int d^3x' \ ([\partial_m T^{mi}_{eff}] x'^j x'^k + [\partial_m T^{mj}_{eff}] x'^i x'^k - [\partial_m T^{mk}_{eff} x'^i x'^j) \\ &= \frac{1}{2} \int d^3x' \ (\partial_t T^{0i}_{eff} x'^j x'^k + \partial_t T^{0j}_{eff} x'^i x'^k - \partial_t T^{0k}_{eff} x'^i x'^j). \end{aligned} \tag{12}$$

On the fourth line we integrated by parts and dropped the boundary terms. On the last line we used conservation of the energy-momentum tensor in the form $\partial_t T^{0j}_{eff} = -\partial_m T^{mj}_{eff}$. Using the fact that $t^{0i} \ll T^{0i}$ we have

$$S^{ijk} = \frac{1}{2} \partial_t \int d^3x' \ (-g)(T^{0i} x'^j x'^k + T^{0j} x'^i x'^k - T^{0k} x'^i x'^j). \tag{13}$$

Combining (7), (11), and (13) we get

$$\begin{aligned} h^{ij}_{TT} =& \frac{4}{x} \Big[\partial_t \int d^3x' \ (-g) T^{0j} x'^i \\ & + n_k \frac{1}{2} \partial_t^2 \int d^3x' \ (-g)(T^{0i} x'^j x'^k + T^{0j} x'^i x'^k - T^{0k} x'^i x'^j) \Big]^{TT}_{t-x} \ . \end{aligned} \tag{14}$$

Since h^{ij} involves only integrals over the matter source T^{ij}, this formulation of the metric perturbation is appropriate to include in computer simulations. Since the quadrupole moment was already implemented in the simulation, I only added the octupole moment (13) to the code. In order to actually implement the octupole-order moment, we must perform the TT projection and choose an energy-momentum tensor.

In order to facilitate taking the TT projection of the octupole moment let us define

$$Q^{ij} = n_k \int d^3x' \ (-g)(T^{0i} x'^j x'^k + T^{0j} x'^i x'^k - T^{0k} x'^i x'^j) \tag{15}$$

so that the octupole moment contribution to h^{ij}_{TT} is $2x^{-1} \partial_t^2 [Q^{ij}]^{TT}_{t-x}$. If the source is at the origin of a spherical coordinate system and the observer is located at (x, θ, ϕ) where θ is the angle down from the z-axis, then the contribution to h_+ and $h_\times$ from the octupole-order moment are given in terms of Q^{ij} by

$$\begin{aligned} h_+ = \frac{1}{x}\partial_t^2[&Q^{11}(\cos^2\theta\cos^2\phi - \sin^2\phi) \\ &+ Q^{12}\sin 2\phi(1+\cos^2\theta) \\ &- Q^{13}\sin 2\theta\cos\phi \\ &+ Q^{22}(\cos^2\theta\sin^2\phi - \cos^2\phi) \\ &- Q^{23}\sin 2\theta\sin\phi \\ &+ Q^{33}\sin^2\theta] \end{aligned} \tag{16}$$

and

$$\begin{aligned} h_\times = \frac{1}{x}\partial_t^2[&-Q^{11}\cos\theta\sin 2\phi \\ &+ 2Q^{12}\cos\theta\cos 2\phi \\ &+ 2Q^{13}\sin\theta\sin\phi \\ &+ Q^{22}\cos\theta\sin 2\phi \\ &- 2Q^{23}\sin\theta\cos\phi] \end{aligned} \tag{17}$$

where $n^k = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

Lastly, we need to specify the energy-momentum tensor $T^{\mu\nu}$ of the source. Following [1], [5], and [6] we treat the star as a fluid and set

$$T^{\mu\nu} = \rho h u^\mu u^\nu + P g^{\mu\nu} \tag{18}$$

where $h = 1+\epsilon+P/\rho$ is the specific enthalpy, ϵ is the specific internal energy of the fluid, P is the pressure of the fluid, ρ is the rest mass density of the fluid, and u^μ is the four velocity. Since matter sources generate more gravity than pressures or energy densities we may rewrite the energy-momentum tensor neglecting P and ϵ. Again following [1], [5], and [6] we set $g = -1$ and choose

$$T^{\mu\nu} = \sqrt{\gamma} W \rho u^\mu u^\nu \tag{19}$$

where $u^\mu = (1, \sqrt{\gamma^{11}}v^1, \sqrt{\gamma^{22}}v^2, \sqrt{\gamma^{33}}v^3)$, γ^{ij} is the three-metric, γ is the determinant of the three-metric, v^i is the three-velocity, and W is the Lorentz factor $(1-v^2)^{-1/2}$.

4. Excited TOV star test simulation

The simulations performed by the Ott group use a collection of in-house and open-source code based on the EinsteinToolkit [7]. After including the octupole order correction in the simulation code, I ran the update simulation on a simple test case. However, as we will see shortly, the test gave relatively little information about the octupole contribution. The test consisted of simulating a Tolman-Oppenheimer-Volkoff (TOV) star where the star's density was perturbed slightly in the shape of a $l = 2$, $m = 0$ spherical harmonic. The simulation was originally developed by the Ott group in [8]. Gravitational wave extraction was performed using the Regge-Wheeler-Zerilli-Moncrief curvature method, the quadrupole moment, and the octupole moment. Figure 1 shows

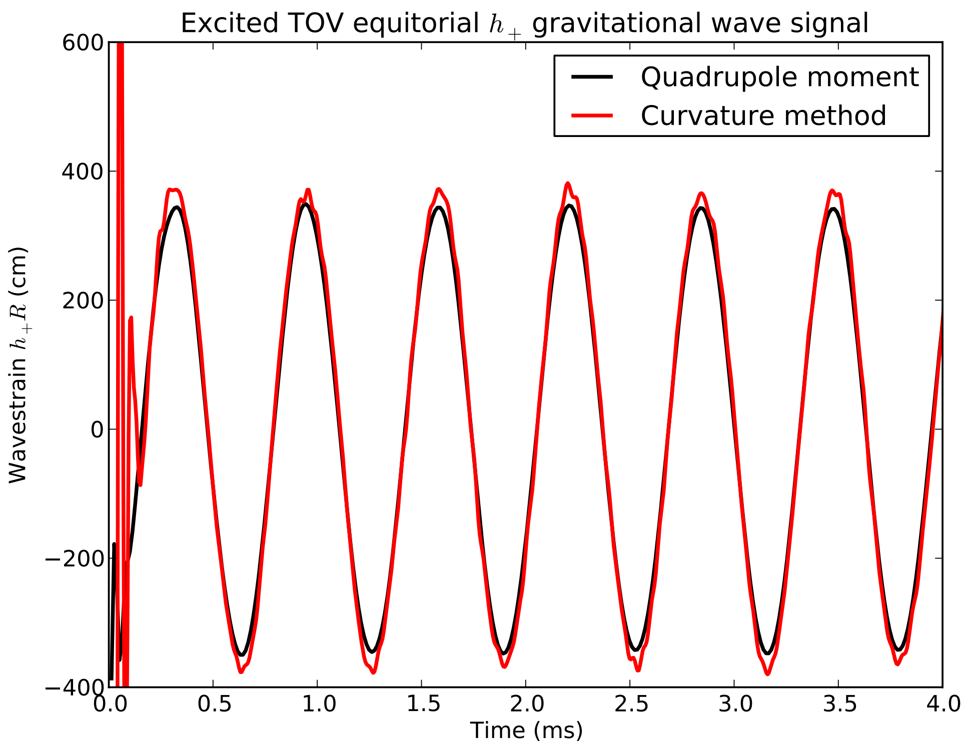

Figure 1. Plot of the gravitational wave emission h_+ at the equator of the perturbed TOV star. The vertical axis is h_+R where R is the distance to the observer. The large gravitational wave signal predicted by the Regge-Wheeler-Zerilli-Moncrief curvature method at early times is due to initial numerical instabilities and is unphysical. There appears to be a scaling factor difference between the two extraction methods that may be due to the choice of velocity variable in the quadrupole moment.

the plus polarization of the gravitational wave signal as a function of time at the equator of the perturbed TOV star. The graph shows good agreement between the curvature method and the quadrupole moment except at early times where the curvature method predicts an unphysically large gravitational wave signal due to numerical errors in the initial conditions. At later times, the two methods appear to differ by roughly a proportionality constant which may be due to the choice of four velocity variable present in the implementation of the quadrupole moment.

While this simulation did test the differences between the quadrupole and Regge-Wheeler-Zerilli-Moncrief extraction methods, it did not yield much information about the octupole moment. Because the density perturbation is a $l = 2$, $m = 0$ (quadrupolar) spherical harmonic, the octupole moment is suppressed. While h_+R was of order 10^2cm for the quadrupole moment (where R is the distance to the observer), h_+R for the octupole moment was noisy and never exceeded 10^{-2}cm. Clearly, the implementation of the octupole moment needs to be tested on a less symmetric setup.

5. Conclusion and further work

In this report we derived an expression (7) for the metric perturbation correct to octupole order. Then we re-expressed the quadrupole and octupole moments in forms more suitable for simulation, see equation (14). The octupole moment was implemented in the Ott group simulations. The quadrupole moment was then compared to the Regge-Wheeler-Zerilli-Moncrief extraction method in the simple case of an excited TOV star. However, the test simulation did not yield much information about the octupole moment as the symmetry of the simulation suppressed that moment. The implementation of the octupole moment can be tested on other less symmetric TOV stars. After testing the octupole moment, we may move on to more computationally expensive simulations like core collapse supernovae.

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Appendix

In this appendix we wish to extract the two leading terms in the multipolar expansion of h^{ij}. We begin with (6) which we repeat here for convenience:

$$h^{ij}(t,\mathbf{x}) = 4\int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|} T^{ij}_{eff}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}') \tag{A.1}$$

We may expand $|\mathbf{x}-\mathbf{x}'|$ and $|\mathbf{x}-\mathbf{x}'|^{-1}$ in terms x'/x which is small because the size of the source is much smaller than the distance between the source and the observer. Since we want h^{ij} correct to two terms, we should expand $|\mathbf{x}-\mathbf{x}'|$ and $|\mathbf{x}-\mathbf{x}'|^{-1}$ to order x'^2/x^2. To perform these expansions, we recall that

$$|\mathbf{x}-\mathbf{x}'|^2 = x^2 - 2\mathbf{x}\cdot\mathbf{x}' + x'^2 = x^2\left[1 - 2\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\left(\frac{x'}{x}\right) + \mathcal{O}\left(\frac{x'}{x}\right)^2\right]. \tag{A.2}$$

To second order the binomial approximations are $(1+\varepsilon)^{\pm 1/2} = 1 \pm \frac{1}{2}\varepsilon + \mathcal{O}(\varepsilon^2)$. Applying the binomial approximations with $\varepsilon = -2\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'(x'/x)$ we get

$$|\mathbf{x}-\mathbf{x}'| = x - x\;\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\left(\frac{x'}{x}\right) + x\;\mathcal{O}\left(\frac{x'}{x}\right)^2 \tag{A.3}$$

and

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{x} + \frac{1}{x}\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\left(\frac{x'}{x}\right) + \frac{1}{x}\mathcal{O}\left(\frac{x'}{x}\right)^2. \tag{A.4}$$

We may now expand $|\mathbf{x}-\mathbf{x}'|^{-1}T^{ij}_{eff}(t-|\mathbf{x}-\mathbf{x}'|)$ to order x'^2/x^2 as

$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{x}'|}T^{ij}_{eff}\left(t-|\mathbf{x}-\mathbf{x}'|\right) &= \left\{\frac{1}{x}+\frac{1}{x}\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\left(\frac{x'}{x}\right)+\frac{1}{x}\mathcal{O}\left(\frac{x'}{x}\right)^2\right\} \\ &\quad\times T^{ij}_{eff}\left(t-x+x\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\left(\frac{x'}{x}\right)+x\mathcal{O}\left(\frac{x'}{x}\right)^2\right).\end{aligned} \tag{A.5}$$

We Taylor expand of T^{ij}_{eff} around $t-x$ to order x'^2/x^2 where the small parameter in the expansion is $x\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\left(\frac{x'}{x}\right)+x\mathcal{O}\left(\frac{x'}{x}\right)^2$ to get

$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{x}'|}T^{ij}_{eff}\left(t-|\mathbf{x}-\mathbf{x}'|\right) &= \left\{\frac{1}{x}+\frac{1}{x}\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\left(\frac{x'}{x}\right)+\frac{1}{x}\mathcal{O}\left(\frac{x'}{x}\right)^2\right\} \\ &\quad\times\left\{T^{ij}_{eff}(t_r)+\left(x\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\left(\frac{x'}{x}\right)+x\mathcal{O}\left(\frac{x'}{x}\right)^2\right)\partial_t T^{ij}_{eff}(t_r)\right. \\ &\quad\left.+\,\mathcal{O}\left(x\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\left(\frac{x'}{x}\right)+x\mathcal{O}\left(\frac{x'}{x}\right)^2\right)^2\right\} \\ &= \frac{1}{x}T^{ij}_{eff}(t_r)+\left(\frac{x'}{x}\right)\left[\frac{1}{x}\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'T^{ij}_{eff}+\frac{1}{x}x\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\partial_t T^{ij}_{eff}(t_r)\right]+\mathcal{O}\left(\frac{x'}{x}\right)^2 \\ &= \frac{1}{x}T^{ij}_{eff}(t_r)+\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\frac{1}{x}\left(\frac{x'}{x}\right)\left[T^{ij}_{eff}(t_r)+x\partial_t T^{ij}_{eff}(t_r)\right]+\mathcal{O}\left(\frac{x'}{x}\right)^2\end{aligned} \tag{A.6}$$

where $t_r=t-x$. We know that $\partial_t T^{ij}_{eff}$ is of order $T^{ij}_{eff}/\Delta t$ where Δt is the time it takes for light to cross the source. Hence, $c\Delta t=\Delta t$ is of order x' so that $x\partial_t T^{ij}_{eff}\sim x/x' T^{ij}_{eff}$. Since $x/x'\gg 1$ we conclude that

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}T^{ij}_{eff}\left(t-|\mathbf{x}-\mathbf{x}'|\right)=\frac{1}{x}T^{ij}_{eff}(t_r)+\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\frac{1}{x}\left(\frac{x'}{x}\right)x\partial_t T^{ij}_{eff}(t_r)+\mathcal{O}\left(\frac{x'}{x}\right)^2 \tag{A.7}$$

Inserting this into the expression for h^{TT}_{ij} we obtain

$$h^{ij}_{TT}=4\int d^3x'\ \left[\frac{1}{x}T^{ij}_{eff}(t_r)+\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\frac{1}{x}\left(\frac{x'}{x}\right)x\partial_t T^{ij}_{eff}(t_r)+\mathcal{O}\left(\frac{x'}{x}\right)^2\right]^{TT} \tag{A.8}$$

which is equivalent to (7). The first term in the integral represents the mass quadrupole moment while the second term in the integral represents the contributions from the mass octupole and current quadrupole moments.

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