Scaling up coating Brownian noise measurements

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1 Intro

The purpose of this document is to calculate levels of coating Brownian thermal noise given laboratory direct measurements at smaller scales. This was partially inspired by this alog post, where some unexpected 'thermal looking' noise was seen, and we'd like to demonstrate that it is not coating Brownian thermal noise. We will do this by applying some scaling rules derived from coating Brownian noise theory. By the end of this document, we should have calculated the level of coating Brownian thermal noise expected in the aLIGO interferometers if they were coated with the same coatings as measured in the TNI [1, 2, 3] and in the Caltech rigid cavity measurement (TNC) [4, 5], but with a film thickness scaled to be comparable with the HR films on the ETMs and ITMs in aLIGO. We can then use these values to compare with the noise levels seen the the aLIGO interferometers to determine if they have reached coating thermal noise limitations.

2 Lab Measurements

2.1 TNI

The bulk of the information on the TNI can be found in reference [1]. However, I had to dig up the mirror dimensions from the DCC [6]. The mirrors are 4" in diameter and 4" in height. The beam waist, $w0$, was about 160 mum for all measurements. Four coatings were measured in the TNI:

1. Quarter-wave stack: silica/tantala
2. Thermal noise optimized stack: silica/tantala
3. Quarter-wave stack: silica/Ti:tantala
4. Thermal noise optimized dichromic stack: silica/Ti:tantala.

These are listed as TNI(1-4) in table 1. All are on silica substrates

The values for $S_x(100\text{Hz})$ were calculated by taking the values for ϕ_{SiO_2}, $\phi_{Ta_2O_5}$, and $\phi_{TiO_2:Ta_2O_5}$ given in tables IV and V of [1], and plugging them

Optic	w_0 [μm]	a [cm]	d [μm]	H [cm]	S_x(100Hz) [m^2 Hz^{-1}]
TNI(1)	160	5.08	4.55	10.16	1.06e-35
TNI(2)	160	5.08	5.41	10.16	8.82e-36
TNI(3)	160	5.08	4.21	10.16	7.79e-36
TNI(4)	160	5.08	3.81	10.16	6.84e-36
TNC	182	1.27	4.53	0.635	1.1e-35
ITM	53000	17	2.8	20	–
ETM	62000	17	5.9	20	–

Table 1: Values useful for converting coating Brownian noise. TNI(1-4) refers to the four different coatings measured in the Thermal Noise Interferometer at Caltech [1, 2, 3]. TNC refers to the reference cavity measurements made at Caltech by Tara Chalermsongsak [4, 5]. Values for ITM and ETM are for the LIGO Input Test Mass and End Test Mass, respectively [7, 8]

back into equations (1) and (2) of that same paper. This should give a good approximation of the actual value of S_x they would have measured.

2.2 TNC

These measurements are nicely covered in [4], with the exception of the substrate thickness, H, which had to be found in [5]. The silica substrates were 1” in diameter and 1/4” in thickness. The film measured here was an old quarter-wave stack produced by REO, made of silica/tantala. To get the value of S_x(100Hz), I plugged their measured value of $\phi_c = 4.43 \times 10^{-4}$ into their equation (8). Again, this should give an approximation of the coating Brownian noise they would have measured at 100 Hz.

3 The simplest scaling

The easiest approach we can do is to look at how scaling works for half-infinite mirrors. In this case, $S_\mathrm{x} \propto d/w^2$ [9]. Therefore, scaling can easily be accomplished by using the following equation:

$$S_\mathrm{x}^\mathrm{pred} = \frac{d_\mathrm{pred}}{d_\mathrm{meas}} \left(\frac{w_\mathrm{meas}}{w_\mathrm{pred}} \right)^2 S_\mathrm{x}^\mathrm{meas}. \tag{1}$$

Here, $S_\mathrm{x}^\mathrm{pred}$ is what we’re after for either ITM or ETM, and d_pred and w_pred would be the coating thickness and beam spot size for the relevant optic (ITM or ETM). Similarly, d_meas and w_meas would be the film thickness and beam spot size for the measured optic (TNI, TNC), and of course, $S_\mathrm{x}^\mathrm{meas}$ would be the directly measured coating Brownian thermal noise.

If we do that, we get the results presented in table 2 and the noise plotted in figure 1. Finally, to turn displacement noise, S_x, for a single mirror into strain

Optic	$\boldsymbol{S^{ITM}_{\text{pred}}}$ **[m^2 Hz^{-1}]**	$\boldsymbol{S^{ETM}_{\text{pred}}}$ **[m^2 Hz^{-1}]**	$\boldsymbol{S_{\text{IFO}}}$ **[Hz^{-1}]**
TNI(1)	5.94e-41	9.15e-41	1.89e-47
TNI(2)	4.16e-41	6.41e-41	1.32e-47
TNI(3)	4.72e-41	7.27e-41	1.50e-47
TNI(4)	4.58e-41	7.05e-41	1.45e-47
TNC	8.02e-41	1.23e-41	2.55e-47

Table 2: Calculated values of $S_x(100\text{Hz})$ for End Test Masses (ETM) and Input Test Masses (ITM), as predicted by scaling the measured noises for each experiment the simple scaling formula in equation 1.

noise in the interferometer, we just add the noise for each mirror and divide by the arm length:

$$S_{\text{IFO}} = 2 * (S_x^{\text{ITM}} + S_x^{\text{ETM}})/L^2, \tag{2}$$

where S_{IFO} is the total power spectral density from mirror Brownian thermal noise in the interferometer, S_x^{ITM} and S_x^{ETM} are the displacement noise we've calculated for the ITM and ETM, respectively, and L is the 4 km arm length of aLIGO. This gives us the final column of table 2. For comparison, the value of $\sqrt{S_{\text{IFO}}}$ from GWINC is 4.25e-24 Hz$^{-1/2}$, or S_{IFO} = 1.81e-47 Hz^{-1}.

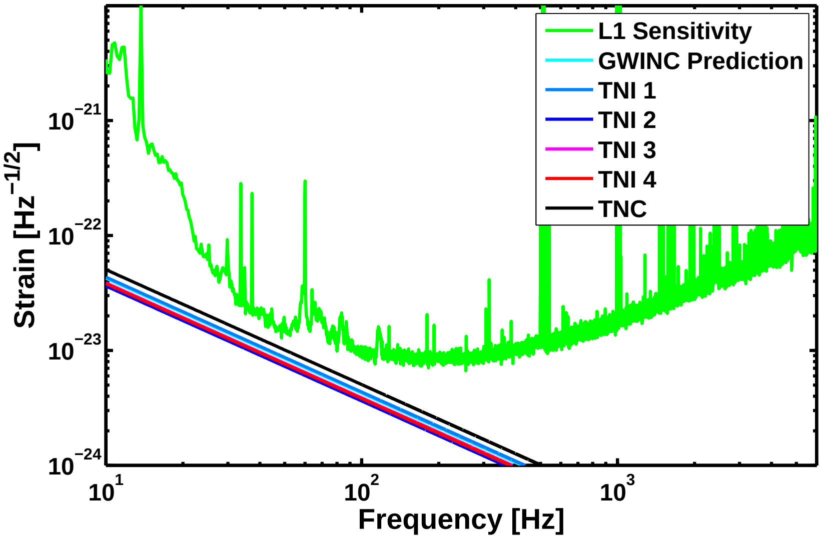

Figure 1: Most recent public sensitivity curve for L1 compared to scaled sensitivity of direct thermal noise measurements using the simple scaling relation in equation 1.

4 Finite Mirror Corrections

The simple scaling in the above section appears to give a reasonable estimation for the expected thermal noise, but it is missing one minor correction; the fact the the mirror dimensions are not the same. The simple scaling assumes that both the laboratory measurement and the aLIGO mirrors are half-infinite. In order to correct for the differences in the physical sizes of the mirrors, we use one of two correction coefficients, those from Liu and Thorne (LaT) [10], or those from Somiya and Yamamoto (SaY) [11]. We'll do both and compare at the end.

In either case, the correction is easy to apply. LaT leads the way by giving a correction factor,

$$(C^{\mathrm{ftm}})^2 = \frac{S^{\mathrm{ftm}}}{S^{\mathrm{itm}}}, \tag{3}$$

S^X is the power spectral density of the coating thermal noise, and the 'ftm' and 'itm' stand for finite test mass and infinite test mass calculations, respectively. Both LaT and SaY give methods for calculating this ratio.

Once we have $(C^{\mathrm{ftm}})^2$, we can adjust the scaling from equation 1 by adding the ratio of correction factors:

$$S_{\mathrm{x}}^{\mathrm{pred}} = \left(\frac{C_{\mathrm{pred}}^{\mathrm{ftm}}}{C_{\mathrm{meas}}^{\mathrm{ftm}}}\right)^2 \frac{d_{\mathrm{pred}}}{d_{\mathrm{meas}}} \left(\frac{w_{\mathrm{meas}}}{w_{\mathrm{pred}}}\right)^2 S_{\mathrm{x}}^{\mathrm{meas}}. \tag{4}$$

4.1 Liu and Thorne

LaT have derived a method of calculating the Brownian thermal noise for a finite-sized substrate and compare that to the half-infinite case. You might think that this is not applicable to the coating thermal noise, but both Nakagawa [9] and Levin [12] have shown (for infinite substrates, anyway), the loss of a total coated substrate is:

$$S^{\mathrm{Total}} = S^{\mathrm{Substrate}} \left[1 + \frac{2}{\sqrt{\pi}} \frac{(1-2\sigma)}{(1-\sigma)} \frac{\phi_{\mathrm{coating}}}{\phi_{\mathrm{substrate}}} \left(\frac{d}{w}\right)\right]. \tag{5}$$

So it's not terribly unreasonable to assume that the only correction we need to make is one to $S^{\mathrm{Substrate}}$, since it's going to get carried through anyway.

Table 3 gives the values of $(C^{\mathrm{ftm}})^2$ calculated using the LaT calculation. These values agree with the same calculation in GWINC for the substrate thermal noise correction. Since the LaT calculation doesn't depend on the film properties, all the TNI measurements get the same correction. When these correction factors are included into equation 5, we get the values in table 4 and figure 2.

4.2 Somiya and Yamamoto

SaY basically did what LaT did but explicitly for coatings. This suggests that their calculations would be more appropriate in this situation. The reason I

Optic	$\boldsymbol{C^{\mathrm{ftm}})^2}$
TNI	0.9973
TNC	1.0259
ITM	0.7099
ETM	0.6610

Table 3: Values useful for converting coating Brownian noise, as calculated using Liu and Thorne [10]. All TNI measurements have the same value because they all used the same substrate size and beam size.

Optic	$\boldsymbol{S^{ITM}_{\mathrm{pred}}}$ **[m^2 Hz^{-1}]**	$\boldsymbol{S^{ETM}_{\mathrm{pred}}}$ **[m^2 Hz^{-1}]**	$\boldsymbol{S_{\mathrm{IFO}}}$ **[Hz^{-1}]**
TNI(1)	4.23e-41	6.07e-41	12.87e-48
TNI(2)	2.96e-41	4.25e-41	9.01e-48
TNI(3)	3.36e-41	4.82e-41	10.22e-48
TNI(4)	3.26e-41	4.68e-41	9.92e-48
TNC	5.55e-41	7.95e-41	16.88e-48

Table 4: Calculated values of S_x(100Hz) for End Test Masses (ETM) and Input Test Masses (ITM), as predicted by scaling the measured noises for each experiment using Liu and Thorne and equation 5.

included LaT above is that, to the best of my knowledge, nobody has ever actually used SaY before. It is not in GWINC, it's not used in any papers that I've read, and some of their plots appear to have some unit errors. In any case, I was able to write up some MatLab code[1] that would do the calculation and the code agrees with at least one of the calculations in SaY's paper. It does not agree with the figures in their paper, but I suspect that has to do with the already-stated unit errors.

Table 5 gives the values of $(C^{\mathrm{ftm}})^2$ calculated using the SaY calculation. It's interesting that these values are much closer to one than the LaT calculations. When these correction factors are included into equation 5, we get the values in table 6 and figure 3.

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[1] You can find it in the DCC, as an auxiliary file to this one.

Optic	$\boldsymbol{C^{\mathrm{ftm}})^2}$
TNI 1	0.9995
TNC	1.0057
ITM	0.9691
ETM	0.9534

Table 5: Values useful for converting coating Brownian noise, as calculated using Somiya and Yamamoto. The TNI values are all the same here, since the variation in film thickness does not affect the results at this precision. [11].

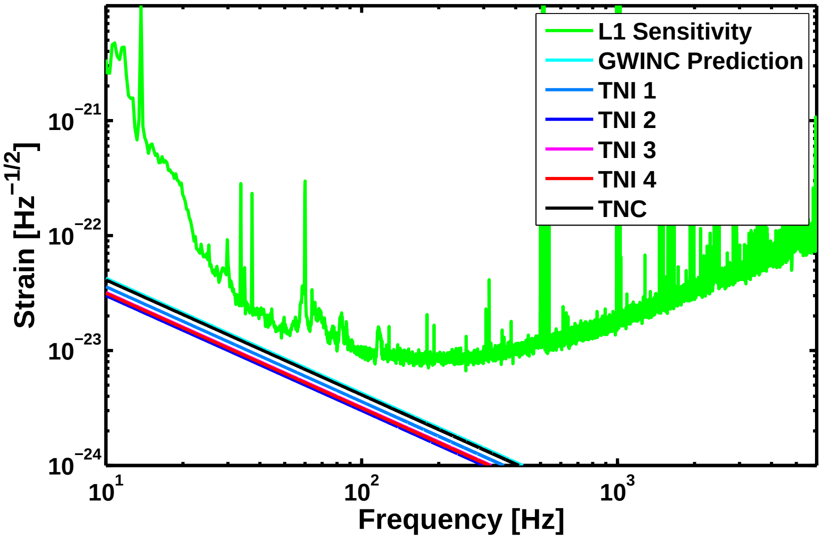

Figure 2: Most recent public sensitivity curve for L1 compared to scaled sensitivity of direct thermal noise measurements using the Liu and Thorne scaling relation and equation 5.

5 Comparison

Looking at figures 1, 2, and 3, it is very difficult to see what difference the finite-sized mass correction makes. In order to see this more clearly, we can look at figure 4 to see the various lines for the scaled measurements of TNI 4, which is the most aLIGO-like coating[2]. This shows that the SaY scaling is very small. LaT makes a fairly large difference, but is mostly likely to be wrong, since it's for substrates. This basically says that simple scaling is good enough and much faster, so we can stick with that. Likely, all three are with uncertainties in the

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[2] Ti:tantala/silica, optimized for red and green light

Optic	$S_{\mathbf{pred}}^{ITM}$ **[m^2 Hz^{-1}]**	$S_{\mathbf{pred}}^{ETM}$ **[m^2 Hz^{-1}]**	$S_{\mathbf{IFO}}$ **[Hz^{-1}]**
TNI(1)	4.23e-41	6.07e-41	12.87e-48
TNI(2)	2.96e-41	4.25e-41	9.01e-48
TNI(3)	3.36e-41	4.82e-41	10.22e-48
TNI(4)	3.26e-41	4.68e-41	9.92e-48
TNC	5.55e-41	7.95e-41	16.88e-48

Table 6: Calculated values of S_x(100Hz) for End Test Masses (ETM) and Input Test Masses (ITM), as predicted by scaling the measured noises for each experiment using Somiya and Yamamoto and equation 5.

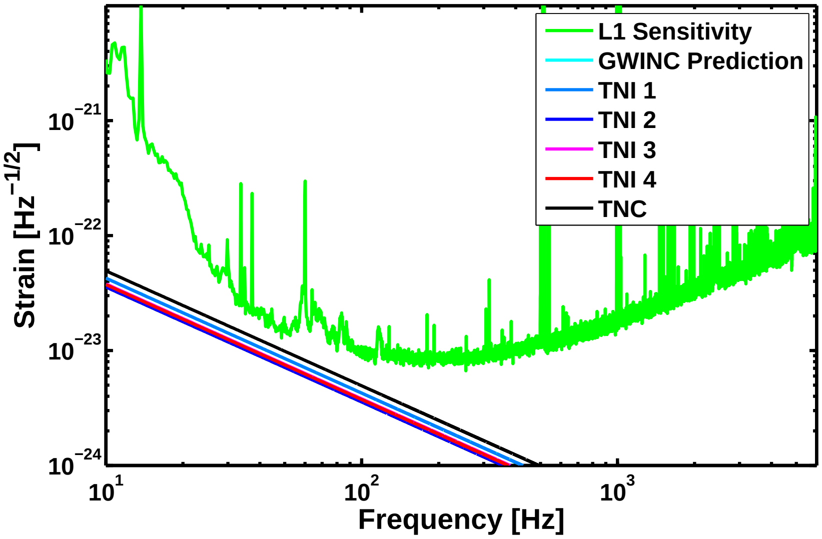

Figure 3: Most recent public sensitivity curve for L1 compared to scaled sensitivity of direct thermal noise measurements using the Somiya scaling relation and equation 5.

measurements.

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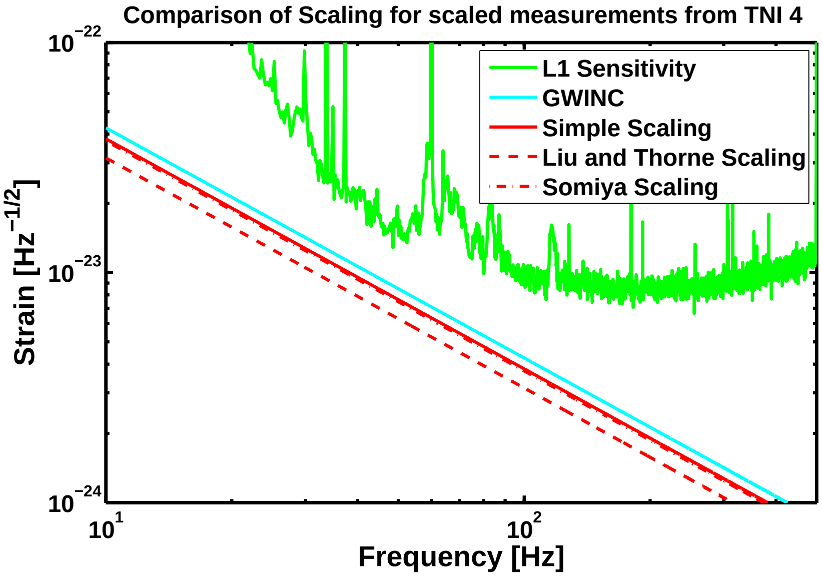

Figure 4: Comparison of the Brownian coating thermal noise scaled using simple scaling, LaT, and SaY.

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