Structure of black holes in theories beyond general relativity
LIGO Final Report

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Abstract

So far in scientific history there has not been a single definitive experiment that shows deviations from the predictions of general relativity (GR). However we know from quantum field theory that GR as it stands is not renormalizable, so it cannot be the full story. LIGO is our first shot at probing gravity in the strong field regime. In order to detect agreement with GR or lack thereof more sensitively, we should generate waveform templates from alternative theories of gravity. This has not been done yet, but by and large we need some more insight from numerical relativistic simulations to develop such prescriptions.

For numerous reasons, we expect these more correct alternative theories to be corrections to GR that have to converge with GR in limiting cases. Therefore it is useful to study spacetimes that are perturbations of that of GR. In our case, we study perturbations of Kerr black holes, the unique class of stationary, axisymmetric, charge neutral, regular, and asymptotically-flat spacetime in GR, with corrections to GR in a theory-independent manner. Our scheme is to study these so called "bumpy black holes" in the Weyl-Lewis-Papapetrou gauge, where the four degrees of freedom of the metric are manifest, and numerically solve for the general class of black holes in beyond-GR theories.

Contents

1 Introduction to beyond GR theories

1.1 Why corrections to GR?

Consider the statement:

$$\textit{There has not been a single experiment that shows disagreement with general relativity.} \tag{1.1}$$

One way to proceed from this fact is to be content with our knowledge of gravity, but another way is to look to investigate this apparent miracle of science.

First of all, the statement is actually not as powerful as one might initially think at a glance. General relativity (GR) is not the only theory consistent with direct observation. And since in science we can only disprove theories, the statement (1.1) is empty except in expressing that we have not eliminated GR from the list of possible theories.

Furthermore, we know that GR is incomplete as a theory. Due to various incompatibility issues, GR is not quantizable as a quantum field theory. The details of these incompatibilities are beyond the scope of this project, but they include the Weinberg-Witten theorem, Lovelock theorem, and non-renormalizability. In short, GR has to break down at sufficiently high energies. We should consider ourselves fortunate that GR has explained and predicted as much as it has in our direct experience so far, but it is not necessarily the be-all and end-all of gravity theories.

However, the success of GR so far does hint at one outlook. Imagine there existed a full (i.e. UV-complete) theory of gravity that is compatible with quantum mechanics. We expect to be able to expand the theory around low energies, integrating out high energy degrees of freedom, to yield an effective field theory (EFT). Because of our century of observations with GR, we expect GR to be the leading order term in our EFT, with higher order corrections appearing. Treating GR in this EFT framework we can consider leading order *perturbations* of GR that become non-negligible at high energies. There has been many theories of gravity that reduce to GR plus perturbation in this limit, and modern EFT effort has rich and fruitful categorizing these theories.

With the inception of the first gravitational wave detectors, such as LIGO, we are entering the era when we can finally probe gravity in the strong field regime and see the agreement with GR break down. It is exciting times in physics when it is plausible that the next experimentally-refutable breakthrough of the origins of gravity is around the corner.

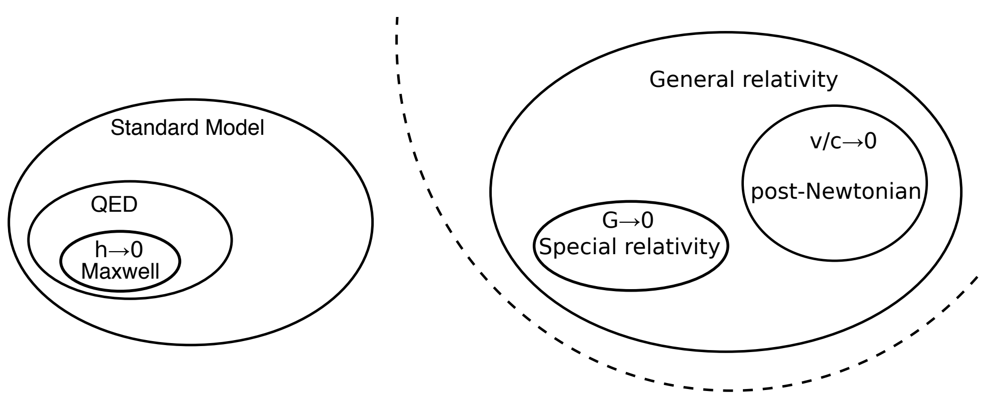


Figure 1: The dotted line shows an next order EFT of gravity with respect to GR and the standard model in "theory" space. Diagram courtesy of Leo C. Stein

2 Motivation

2.1 Tests of GR

There are two distinct ways to study theories of gravity: theory-dependently and theory-independently. Theory-dependent tests are great for statistics and getting detections. We increase our signal-to-noise we are looking for a specific kind of signal, e.g. matched filtering. However, we would like to create a framework were we can test a huge class of theories all at once, where each specific theory is parameterized so that we can better connect observation to new theories.

2.2 Numerical simulation

The broad on-going goal of this project to develop a theory independent framework that could in principle be applied to virtually all known subleading EFTs of gravity. And for demonstration purposes and to generate waveforms that could be use for gravitational wave detectors, subsets or specific theories will be chosen for numerical simulation. We need simulate compact binary merger in these theories to determine what the characteristics effects of on a distant gravitational wave detectors. Since waveforms informed by alternative theories of gravity has not been implemented in the search pipelines of current generation detectors like LIGO and VIRGO, we have good impetus to create such waveforms.

2.3 Perturbations of GR

To recap, from modern effective field theory principles, we can view alternative theories of gravity as *corrections* to GR. This means we can do perturbation theory. Specifically we perform do perturbation theory on gravity theories. Perturbations of the theory will yield in perturbations of GR spacetime solutions.

Before embarking on compact binary merger simulations, we focus our project on stationary and isolated, black holes, where no matter is unmodeled. This is usually represented by the Kerr solution in GR. The program we are developing from ground up should be generalizable to binary black holes and systems with modelled matter.

In this project, we study general axisymmetric, stationary perturbations of the Kerr spacetime, a solution of GR, and develop a formalism for solving perturbations from Kerr. Depending on your point of view, the perturbations can represent corrections to GR, but they can also represent matter near the horizons of black holes, known as "*bumpy black holes*".

Note here that the perturbed spacetime does not have to be Ricci-flat, a property that most known analytic black hole solutions in GR have.

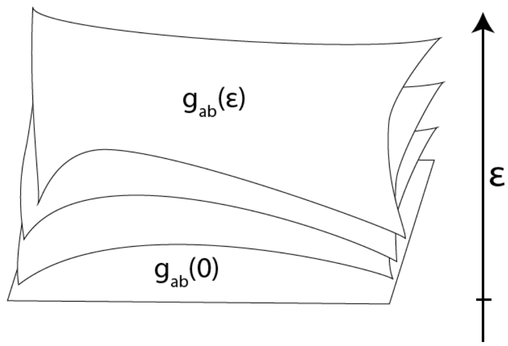
3 Setup

3.1 Perturbation theory

We consider a 1-parameter family of spacetime geometries, described by the metric g_{ab} with the parameter ϵ. Expanding around $\epsilon = 0$ for sufficiently small ϵ,

$$g_{ab}(\epsilon) = g_{ab}{}^{(\epsilon=0)} + \epsilon \underbrace{\left.\frac{dg_{ab}}{d\epsilon}\right|_{\epsilon=0}}_{h^{(1)}_{ab}} + \mathcal{O}(\epsilon^2) \tag{3.1}$$

In our prescription, let $g_{ab}{}^{(\epsilon=0)}$ be a Ricci-flat geometry like Schwarzschild or Kerr.



3.2 Theory independent Lagrangian

For example, the action for a general interacting scalar θ non-minimally coupled to curvature

$$I = \int d^4x\sqrt{-g}\left[R - \frac{1}{2}\partial_a\theta\partial^a\theta + \epsilon\mathcal{L}_{\text{int}}\right] \tag{3.2}$$

For example, the interaction can be θ coupled to the Pontryagin density in dynamical Chern-Simons theory, $\mathcal{L}_{\text{int}} = \theta\ {}^*RR$. We picked a scalar field for the sake of numerical simulations, but we can easily adapt other fields into our formalism.

3.3 Theory independent Equations of Motion

Detailed in later in section 4.5, we can show that the Einstein tensor, which captures space time curvature, G_{ab} is equal to an effective stress energy tensor T^{eff}_{ab}, where $T^{\text{eff}}_{ab} = \mathcal{O}(\epsilon^2)$ and $\nabla^a T^{\text{eff}}_{ab} = 0$. That is, spacetime curvature is sourced by the energy-momentum of the scalar field and the scalar-curvature interaction.

Additionally the by varying the action with respect to the scalar field, we will get the 4-dimensional laplacian acting on the scalar $\Box\theta$ equal to to a source arising from the variation of the interaction piece of the Lagrangian $S = \mathcal{O}(\epsilon^1)$.

This is a compact way to think about these equations of motion: in a coordinates system called Lorentz gauge, we can write that the two relevant equations of motion for the metric perturbations h_{ab} and scalar θ are $\Box h_{ab} = T^{\text{eff}}_{ab}$ and $\Box\theta = S$, repsectively.

3.4 Gauge for black holes

In gravity, diffeomorphisms can be thought of as coordinate transformations that leave spacetime invariant. So while the physics of gravity remains the same, the coordinate freedom is what we call *gauge freedom*. We will see in a later section (4.4.3) that infinitesimal diffeomorphisms correspond to the gauge transformation not unlike a spin-2 gauge boson.

While it doesn't matter which gauge we choose because the physics remains the same, we must still remain judicious in our choice of gauge. This is because analytical analysis can be greatly elucidated or obscured depending on the gauge. And numerical analysis can plain converge quickly or not converge at all due to our choice of gauge. When we are working with spacetimes that can curve and oscillate, the coordinates can curve and oscillate too, distinguishing between the two is a subtle task that takes care. Gauge becomes not a convenient redundancy but rather a serious issue we must consider at every step of this project.

3.5 Black hole spacetimes

For astrophysical blackholes, we can make a few observations that yield simplifying assumptions. First, conservation of electric charge and the Coulomb force implies that the black hole any charge in the black hole be canceled by opposite charges streaming in. We can take astrophysical blackholes to be essentially electrically uncharged. Similarly, conservation of angular momentum implies that after all the collisions of matter, a black hole settles down into a state with a well defined angular momentum, and into an axisymmetric state, where there exists an aximuthal direction for which the black hole can be modeled to be symmetric. Lastly, at late times of the black hole evolution, we expect all non-stationary parts of the black hole to gravitationally radiate away. So we can consider stationary black holes for now. In this project, we are, in a sense, considering only steady-state solutions of black holes in theories that corrections to GR.

Because of these assumptions, we are motivated to put our metrics into Weyl-Lewis-Papapetrou (WLP) gauge, which includes virtually all stationary, axisymmetric spacetimes. We will explore this in greater detail, but the line element due to this metric is

$$ds^2 = -V(dt - wd\phi)^2 + V^{-1}\left(\rho^2 d\phi^2 + e^{2\gamma}(d\rho^2 + e^{2\lambda}dz^2)\right) \tag{3.3}$$

Note that unlike the Kerr solution in GR, this metric doesn't have to be Ricci-flat.

4 General Spacetime Results

Despite not requiring any explicit coordinates or gauge, following results will be serve as the fundamental language we use for the rest of the project.

4.1 Perturbation Theory

To reiterate, we have

$$g_{ab}(\lambda) = {g_{ab}}^{(0)} + \lambda \left.\frac{dg_{ab}}{d\lambda}\right|_{\lambda=0} + \frac{\lambda^2}{2}\left.\frac{d^2 g_{ab}}{d\lambda^2}\right|_{\lambda=0} + \mathcal{O}(\lambda^3) \tag{4.1}$$

$$\equiv {g_{ab}}^{(0)} + \lambda h^{(1)}_{ab} + \frac{\lambda^2}{2} h^{(2)}_{ab} + \mathcal{O}(\lambda^3) \tag{4.2}$$

where ${g_{ab}}^{(0)}$ is the background Ricci-flat spacetime and $h^{(1)}_{ab} = \left.\frac{dg_{ab}}{d\lambda}\right|_{\lambda=0}$ is the first order metric perturbation.

4.2 Connection on a Background

We have the difference of connections, where $\nabla^{(\lambda)}_a$ is compatible with the metric ${g_{bc}}^{(\lambda)}$:

$$(\nabla^{(\lambda)}_a - \nabla^{(0)}_a)v^b = C^b_{ac}v^c \tag{4.3}$$

$$(\nabla^{(\lambda)}_a - \nabla^{(0)}_a)\omega_b = -C^c_{ab}\omega_c \tag{4.4}$$

where C^c_{ab} is a function of λ.

Therefore, from $0 = \nabla^{(\lambda)}_c {g_{ab}}^{(\lambda)}$, we have two identities:

$$C^c_{ab} = \frac{1}{2}{g^{cd}}_{(\lambda)}\left({\nabla_a}^{(0)}{g_{db}}^{(\lambda)} + {\nabla_b}^{(0)}{g_{ad}}^{(\lambda)} - {\nabla_d}^{(0)}{g_{ab}}^{(\lambda)}\right) \tag{4.5}$$

$$C^c_{ab} = \frac{1}{2}{g^{cd}}_{(\lambda)}\left(\partial_a {g_{db}}^{(\lambda)} + \partial_b {g_{ad}}^{(\lambda)} - \partial_d {g_{ab}}^{(\lambda)}\right) - \frac{1}{2}{g^{cd}}_{(0)}\left(\partial_a {g_{db}}^{(0)} + \partial_b {g_{ad}}^{(0)} - \partial_d {g_{ab}}^{(0)}\right) \tag{4.6}$$

For notational convenience let $\tilde{\nabla}_a \equiv {\nabla_a}^{(0)}$ and $\nabla_a \equiv {\nabla_a}^{(\lambda)}$. The Riemann curvature tensor is

$$\begin{aligned}
{R_{abc}}^d\omega_d &\equiv [\nabla_a, \nabla_b]\omega_c && (4.7)\\
&= \nabla_a\nabla_b\omega_c - (a \leftrightarrow b) && (4.8)\\
&= \tilde{\nabla}_a(\nabla_b\omega_c) - \cancel{C^d_{ab}(\nabla_d\omega_c)} - C^d_{ac}(\nabla_b\omega_d) - (a \leftrightarrow b) && (4.9)\\
&= \tilde{\nabla}_a(\tilde{\nabla}_b\omega_c - C^d_{bc}\omega_d) - C^d_{ac}(\tilde{\nabla}_b\omega_d - C^e_{bd}\omega_e) - (a \leftrightarrow b) && (4.10)\\
&= \tilde{\nabla}_a\tilde{\nabla}_b\omega_c - \tilde{\nabla}_a C^d_{bc}\omega_d \cancel{- C^d_{bc}\tilde{\nabla}_a\omega_d - C^d_{ac}\tilde{\nabla}_b\omega_d} + C^d_{ac}C^e_{bd}\omega_e - (a \leftrightarrow b) && (4.11)\\
&= \tilde{\nabla}_{[a}\tilde{\nabla}_{b]}\omega_c - \tilde{\nabla}_{[a}C^d_{b]c}\omega_d + C^d_{c[a}C^e_{b]d}\omega_e && (4.12)\\
&= \left({R^d}_{abc}{}^{(0)} - \tilde{\nabla}_{[a}C^d_{b]c} + C^e_{c[a}C^d_{b]e}\right)\omega_d && (4.13)
\end{aligned}$$

$$\implies \boxed{{R_{abc}}^d = {R_{abc}}^d{}_{(0)} - \tilde{\nabla}_{[a}C^d_{b]c} + C^e_{c[a}C^d_{b]e}} \tag{4.14}$$

4.3 Linearized Einstein Operator

The linearized Einstein operator is the linearized Einstein tensor acting on the space of perturbations. Specifically

$$G^{(1)} : C^{\infty}\left[\mathrm{Sym}^2(T^*\mathcal{M})\right] \to C^{\infty}\left[\mathrm{Sym}^2(T^*\mathcal{M})\right] \tag{4.15}$$

This operator is possibly a Lichnerowicz operator, but it is unconfirmed. It will be shown in section 4.6.2 that this operator self-adjoint with respect to a natural inner product.

Let $\tilde{\nabla}_a \equiv {\nabla_a}^{(0)}$ and $g_{ab} = {g_{ab}}^{(\lambda)}$ unless otherwise specified.

$$C^c_{ab} = \frac{1}{2} g^{cd}\left(\tilde{\nabla}_a g_{db} + \tilde{\nabla}_b g_{ad} - \tilde{\nabla}_d g_{ab}\right) \tag{4.16}$$

$${C^c_{\ ab}}^{(0)} = 0 \tag{4.17}$$

$$\implies C^c_{ab} = \mathcal{O}(\lambda) \tag{4.18}$$

$$C^c_{ab} = \frac{1}{2}\lambda g^{cd(0)}\left(\tilde{\nabla}_a h_{db} + \tilde{\nabla}_b h_{ad} - \tilde{\nabla}_d h_{ab}\right) + \mathcal{O}(\lambda^2) \tag{4.19}$$

We have

$${R_{abc}}^d = {R_{abc}}^d_{\ (0)} - \tilde{\nabla}_{[a} C^d_{b]c} + \mathcal{O}(\lambda^2) \tag{4.20}$$

$$\implies R_{ac} = {R_{ac}}^{(0)} - \tilde{\nabla}_{[a} C^d_{d]c} + \mathcal{O}(\lambda^2) \tag{4.21}$$

$$= {R_{ac}}^{(0)} - \frac{1}{2}\lambda g^{de(0)}\left(\tilde{\nabla}_a\tilde{\nabla}_d h_{ec} + \tilde{\nabla}_a\tilde{\nabla}_c h_{de} - \tilde{\nabla}_a\tilde{\nabla}_e h_{dc} - (a \leftrightarrow d)\right) + \mathcal{O}(\lambda^2) \tag{4.22}$$

$$= {R_{ac}}^{(0)} - \frac{1}{2}\lambda\left([\tilde{\nabla}_a, \tilde{\nabla}^e] h_{ec} + \tilde{\nabla}_a\tilde{\nabla}_c h - \tilde{\nabla}^e\tilde{\nabla}_c h_{ae} - \tilde{\nabla}_a\tilde{\nabla}^d h_{dc} + \tilde{\nabla}_d\tilde{\nabla}^d h_{ac}\right) + \mathcal{O}(\lambda^2) \tag{4.23}$$

$$\boxed{R_{ac} = {R_{ac}}^{(0)} - \frac{1}{2}\lambda\left(\tilde{\nabla}_a\tilde{\nabla}_c h - \tilde{\nabla}^e\tilde{\nabla}_{(a} h_{c)e} + \tilde{\nabla}_d\tilde{\nabla}^d h_{ac}\right) + \mathcal{O}(\lambda^2)} \tag{4.24}$$

where we have $\tilde{\nabla}_a$ and h_{ab} raised and lowered (and traced) by the background metric ${g^{cd}}_{(0)}$.

Furthermore, we have

$$R = g^{ac} R_{ac} \tag{4.25}$$

$$= (g^{ac(0)} - \lambda h^{ac}){R_{ac}}^{(0)} - \frac{1}{2}\lambda g^{ac(0)}\left(\tilde{\nabla}_a\tilde{\nabla}_c h - \tilde{\nabla}^e\tilde{\nabla}_{(a} h_{c)e} + \tilde{\nabla}_d\tilde{\nabla}^d h_{ac}\right) + \mathcal{O}(\lambda^2) \tag{4.26}$$

$$= R^{(0)} - \lambda h^{ac}{R_{ac}}^{(0)} - \frac{1}{2}\lambda\left(\tilde{\nabla}_a\tilde{\nabla}^a h - 2\tilde{\nabla}^e\tilde{\nabla}^a h_{ae} + \tilde{\nabla}_d\tilde{\nabla}^d h\right) + \mathcal{O}(\lambda^2) \tag{4.27}$$

$$\boxed{R = R^{(0)} - \lambda\left(h^{ac}{R_{ac}}^{(0)} + \tilde{\nabla}_d\tilde{\nabla}^d h - \tilde{\nabla}^c\tilde{\nabla}^d h_{cd}\right) + \mathcal{O}(\lambda^2)} \tag{4.28}$$

Therefore the linearized Einstein tensor is

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab} \tag{4.29}$$

$$= {R_{ab}}^{(0)} - \frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_{(a} h_{b)e} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ab}\right) \tag{4.30}$$

$$- \frac{1}{2}({g_{ab}}^{(0)} + \lambda h_{ab}) \left[R^{(0)} - \lambda \left(h^{cd} {R_{cd}}^{(0)} + \tilde{\nabla}_d \tilde{\nabla}^d h - \tilde{\nabla}^c \tilde{\nabla}^d h_{cd}\right)\right] + \mathcal{O}(\lambda^2) \tag{4.31}$$

$$= {G_{ab}}^{(0)} - \frac{1}{2}\lambda h_{ab} R^{(0)} + \frac{1}{2}\lambda {g_{ab}}^{(0)} h^{cd} {R_{cd}}^{(0)} - \frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_{(a} h_{b)e} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ab}\right) \tag{4.32}$$

$$+ \frac{1}{2}\lambda {g_{ab}}^{(0)} \left(h^{cd} {R_{cd}}^{(0)} + \tilde{\nabla}_d \tilde{\nabla}^d h - \tilde{\nabla}^c \tilde{\nabla}^d h_{cd}\right) + \mathcal{O}(\lambda^2) \tag{4.33}$$

If we have a Ricci-flat background, ${R_{cd}}^{(0)} = 0$,

$$\boxed{G_{ab} = -\frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_{(a} h_{b)e} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ab} - {g_{ab}}^{(0)} \tilde{\nabla}_d \tilde{\nabla}^d h + {g_{ab}}^{(0)} \tilde{\nabla}^c \tilde{\nabla}^d h_{cd}\right) + \mathcal{O}(\lambda^2)} \tag{4.34}$$

which agrees with the Fierz-Pauli equation for massless spin-2 bosons in a Minkowski background.

We can also note that $\lambda \nabla_a = \lambda \tilde{\nabla}_a + \mathcal{O}(\lambda^2)$, so

$$G_{ab} = -\frac{1}{2}\lambda \left(\nabla_a \nabla_b h - \nabla^e \nabla_{(a} h_{b)e} + \nabla_d \nabla^d h_{ab} - g_{ab} \nabla_d \nabla^d h + g_{ab} \nabla^c \nabla^d h_{cd}\right) + \mathcal{O}(\lambda^2) \tag{4.35}$$

4.4 Gauge conditions

4.4.1 Covariant Derivative Commutator derivation

Given that $[\tilde{\nabla}_a, \tilde{\nabla}_b]\omega_c = -{R^d}_{cab}{}^{(0)} \omega_d$, we have

$$[\tilde{\nabla}_a, \tilde{\nabla}_b](h_{cd} v^d) = -{R^e}_{cab}{}^{(0)} (h_{ed} v^d) \tag{4.36}$$

$$\tilde{\nabla}_a \tilde{\nabla}_b h_{cd} v^d + \cancel{\tilde{\nabla}_b h_{cd} \tilde{\nabla}_a v^d + \tilde{\nabla}_a h_{cd} \tilde{\nabla}_b v^d} + h_{cd} \tilde{\nabla}_a \tilde{\nabla}_b v^d - (a \leftrightarrow b) = -{R^e}_{cab}{}^{(0)} (h_{ed} v^d) \tag{4.37}$$

$$[\tilde{\nabla}_a, \tilde{\nabla}_b] h_{cd} v^d + h_{ce} [\tilde{\nabla}_a, \tilde{\nabla}_b] v^e = -{R^e}_{cab}{}^{(0)} (h_{ed} v^d) \tag{4.38}$$

$$[\tilde{\nabla}_a, \tilde{\nabla}_b] h_{cd} v^d + h_{ce} {R^e}_{dab}{}^{(0)} v^d = -{R^e}_{cab}{}^{(0)} h_{ed} v^d \tag{4.39}$$

$$\boxed{[\tilde{\nabla}_a, \tilde{\nabla}_b] h_{cd} = -{R^e}_{cab}{}^{(0)} h_{ed} - {R^e}_{dab}{}^{(0)} h_{ce}} \tag{4.40}$$

4.4.2 Lorenz Gauge of the Trace-reverse of Metric Perturbation

In Lorenz gauge, $0 = \tilde{\nabla}^a \bar{h}_{ab} = \tilde{\nabla}^a h_{ab} - \frac{1}{2} g_{ab} \tilde{\nabla}^a h$ in $3+1$ dimensions with a Ricci-flat background

$$G_{ab} = -\frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_{(a} h_{b)e} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ab} - g_{ab} \tilde{\nabla}_d \tilde{\nabla}^d h + g_{ab} \tilde{\nabla}^c \tilde{\nabla}^d h_{cd} \right) + \mathcal{O}(\lambda^2) \tag{4.41}$$

$$= -\frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_{(a} h_{b)e} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ab} - g_{ab} \tilde{\nabla}_d \tilde{\nabla}^d h + \frac{1}{2} g_{ab} \tilde{\nabla}^c (g_{cd} \tilde{\nabla}^d h) \right) + \mathcal{O}(\lambda^2) \tag{4.42}$$

$$= -\frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_{(a} h_{b)e} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ab} - \frac{1}{2} g_{ab} \tilde{\nabla}_d \tilde{\nabla}^d h \right) + \mathcal{O}(\lambda^2) \tag{4.43}$$

$$= -\frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_a h_{be} - \tilde{\nabla}^e \tilde{\nabla}_b h_{ae} + \tilde{\nabla}_d \tilde{\nabla}^d \bar{h}_{ab} \right) + \mathcal{O}(\lambda^2) \tag{4.44}$$

$$= -\frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_a \left(\bar{h}_{be} + \frac{1}{2} g_{be} h \right) - \tilde{\nabla}^e \tilde{\nabla}_b \left(\bar{h}_{ae} + \frac{1}{2} g_{ae} h \right) + \tilde{\nabla}_d \tilde{\nabla}^d \bar{h}_{ab} \right) + \mathcal{O}(\lambda^2) \tag{4.45}$$

$$= -\frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_a \bar{h}_{be} - \frac{1}{2} \tilde{\nabla}_b \tilde{\nabla}_a h - \tilde{\nabla}^e \tilde{\nabla}_b \bar{h}_{ae} - \frac{1}{2} \tilde{\nabla}_a \tilde{\nabla}_b h + \tilde{\nabla}_d \tilde{\nabla}^d \bar{h}_{ab} \right) + \mathcal{O}(\lambda^2) \tag{4.46}$$

$$= -\frac{1}{2}\lambda \left(-\tilde{\nabla}^e \tilde{\nabla}_{(a} \bar{h}_{b)e} + \tilde{\nabla}_d \tilde{\nabla}^d \bar{h}_{ab} \right) + \mathcal{O}(\lambda^2) \tag{4.47}$$

$$= -\frac{1}{2}\lambda \left(-g^{ec} \left([\tilde{\nabla}_c, \tilde{\nabla}_a] \bar{h}_{be} + (a \leftrightarrow b) \right) + \tilde{\Box} \bar{h}_{ab} \right) + \mathcal{O}(\lambda^2) \tag{4.48}$$

$$= -\frac{1}{2}\lambda \left(-g^{ec} \left(-R^{d}{}_{bca}{}^{(0)} \bar{h}_{de} - R^{d}{}_{eca}{}^{(0)} \bar{h}_{bd} + (a \leftrightarrow b) \right) + \tilde{\Box} \bar{h}_{ab} \right) + \mathcal{O}(\lambda^2) \tag{4.49}$$

$$= -\frac{1}{2}\lambda \left(+ \left(R^{d}{}_{b}{}^{e}{}_{a}{}^{(0)} \bar{h}_{de} + \cancel{R^{d}{}_{e}{}^{e}{}_{a}{}^{(0)} \bar{h}_{bd}} + (a \leftrightarrow b) \right) + \tilde{\Box} \bar{h}_{ab} \right) + \mathcal{O}(\lambda^2) \tag{4.50}$$

$$\boxed{G_{ab} = -\frac{1}{2}\lambda \left(2 R^{c}{}_{a}{}^{d}{}_{b}{}^{(0)} \bar{h}_{cd} + \tilde{\Box} \bar{h}_{ab} \right) + \mathcal{O}(\lambda^2)} \tag{4.51}$$

Note that for WLP gauge that we choose later, $h = 0$, so $\bar{h}_{ab} = h_{ab}$.

4.4.3 Infinitesimal Gauge Transformation

We see that infinitesimal diffeomorphism $x^a \mapsto x'^{a'} = x^{a'} + \kappa^{a'}$, is equivalent to an infinitesimal gauge transformation of the metric at linear order:

$$g^{ab}(x) \mapsto g^{a'b'}(x') \tag{4.52}$$

$$= \frac{\partial x'^{a'}}{\partial x^a} \frac{\partial x'^{b'}}{\partial x^b} g^{ab}(x) \tag{4.53}$$

$$= (\delta_a^{a'} + \partial_a \kappa^{a'})(\delta_b^{b'} + \partial_b \kappa^{b'}) g^{ab}(x) \tag{4.54}$$

$$= \left(\delta_a^{a'} \delta_b^{b'} + \delta_a^{a'} \partial_b \kappa^{b'} + \partial_a \kappa^{a'} \delta_b^{b'} + \mathcal{O}(\kappa^2) \right) g^{ab}(x) \tag{4.55}$$

$$= g^{a'b'}(x) + \partial^{a'} \kappa^{b'} + \partial^{b'} \kappa^{a'} + \mathcal{O}(\kappa^2) \tag{4.56}$$

Therefore for first order perturbations, $h_{ab} \mapsto h_{ab} + \nabla_a^{(0)} \kappa_b + \nabla_b^{(0)} \kappa_a$ is a gauge transformation for arbitrary infinitesimal covector field κ_a. Note, this is exactly the gauge transformation for spin-2 gauge bosons. We see that for the 10 components of h_{ab}, we have 4 gauge degrees of freedom. The remaining 6 are 2 propagating degrees of freedom and 4 static components.

4.5 Decoupling Limit of Scalar field

In the decoupling limit, we have for smalle ϵ, the action for an interacting scalar field (e.g. dynamical Chern-Simons):

$$I = \int d^4x\sqrt{-g}\left[\frac{m_p^2}{2}R - \frac{1}{2}\partial_a\theta\partial^a\theta + \epsilon\mathcal{L}_{\text{int}}\right] \tag{4.57}$$

Imposing the principle of stationary action,

$$\begin{aligned}
0 &= \delta I && (4.58)\\
&= \int\left\{\delta\sqrt{-g}\left[\frac{m_p^2}{2}R - \frac{1}{2}\partial_a\theta\partial^a\theta + \epsilon\mathcal{L}_{\text{int}}\right] + \sqrt{-g}\delta\left[\frac{m_p^2}{2}R - \frac{1}{2}\partial_a\theta\partial^a\theta + \epsilon\mathcal{L}_{\text{int}}\right]\right\}d^4x && (4.59)\\
&= \int d^4x\sqrt{-g}\left\{-\frac{1}{2}g_{ab}\delta g^{ab}\left[\frac{m_p^2}{2}R - \frac{1}{2}\partial_c\theta\partial^c\theta + \epsilon\mathcal{L}_{\text{int}}\right] + \frac{m_p^2}{2}\delta R + \delta\left[-\frac{1}{2}\partial_c\theta\partial^c\theta + \epsilon\mathcal{L}_{\text{int}}\right]\right\} && (4.60)\\
&= \int d^4x\sqrt{-g}\left\{-\frac{1}{2}g_{ab}\delta g^{ab}\left[\frac{m_p^2}{2}R - \frac{1}{2}\partial_c\theta\partial^c\theta + \epsilon\mathcal{L}_{\text{int}}\right] + \frac{m_p^2}{2}R_{ab}\delta g^{ab} - \frac{1}{2}\delta(\partial_c\theta\partial^c\theta) + \delta\left[\epsilon\mathcal{L}_{\text{int}}\right]\right\} && (4.61)\\
&= \int d^4x\sqrt{-g}\delta g^{ab}\left\{\frac{m_p^2}{2}\left(R_{ab} - \frac{1}{2}g_{ab}R\right) - \frac{1}{2}g_{ab}\left[-\frac{1}{2}\partial_c\theta\partial^c\theta + \epsilon\mathcal{L}_{\text{int}}\right]\right. && (4.62)\\
&\quad \left.-\frac{1}{2}\frac{\delta}{\delta g^{ab}}(g^{cd}\partial_c\theta\partial_d\theta) + \frac{\delta}{\delta g^{ab}}(\epsilon\mathcal{L}_{\text{int}})\right\} + \int d^4x\sqrt{-g}\left\{-\frac{\frac{1}{2}\delta(\partial_c\theta\partial^c\theta)}{\delta\theta}\delta\theta + \epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\delta\theta\right\} && (4.63)\\
&= \int d^4x\sqrt{-g}\delta g^{ab}\left\{\frac{m_p^2}{2}G_{ab} - \frac{1}{2}g_{ab}\left[-\frac{1}{2}\partial_c\theta\partial^c\theta + \epsilon\mathcal{L}_{\text{int}}\right] - \frac{1}{2}\delta_a^c\delta_b^d\partial_c\theta\partial_d\theta + \epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right\} && (4.64)\\
&\quad + \int d^4x\sqrt{-g}\left\{-\frac{\partial_c\theta\partial^c\delta\theta}{\delta\theta}\delta\theta + \epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\delta\theta\right\} && (4.65)\\
&= \frac{1}{2}\int d^4x\sqrt{-g}\delta g^{ab}\left\{m_p^2G_{ab} - \left[\partial_a\theta\partial_b\theta - \frac{1}{2}g_{ab}\partial_c\theta\partial^c\theta\right] + \epsilon\mathcal{L}_{\text{int}}g_{ab} + 2\epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right\} && (4.66)\\
&\quad + \int d^4x\delta\theta\left\{+\partial^c(\sqrt{-g}\nabla_c\theta) + \epsilon\sqrt{-g}\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right\} && (4.67)\\
0 &= \frac{1}{2}\int d^4x\sqrt{-g}\delta g^{ab}\left\{m_p^2G_{ab} - \left[\partial_a\theta\partial_b\theta - \frac{1}{2}g_{ab}\partial_c\theta\partial^c\theta\right] + \epsilon\mathcal{L}_{\text{int}}g_{ab} + 2\epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right\} && (4.68)\\
&\quad + \int d^4x\sqrt{-g}\delta\theta\left\{+\nabla^c\nabla_c\theta + \epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right\} && (4.69)
\end{aligned}$$

Therefore our equations of motion are:

$$\boxed{\begin{aligned}
{m_p}^2G_{ab} + \underbrace{\epsilon\mathcal{L}_{\text{int}}g_{ab} + 2\epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}}_{\epsilon C_{ab}} &= \underbrace{\partial_a\theta\partial_b\theta - \frac{1}{2}g_{ab}\partial_c\theta\partial^c\theta}_{{T_{ab}}^{(\theta)}} && (4.70)\\
\Box\theta &= \underbrace{-\epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}}_{S} && (4.71)
\end{aligned}}$$

We have the perturbative expansion from a Ricci-flat, scalarless background:

$$\theta = 0 + \epsilon\theta^{(1)} + \frac{1}{2}\epsilon^2\theta^{(2)} + \mathcal{O}(\epsilon^3) \tag{4.72}$$

$$g_{ab} = {g_{ab}}^{(0)} + \epsilon {h_{ab}}^{(1)} + \frac{1}{2}\epsilon^2 {h_{ab}}^{(2)} + \mathcal{O}(\epsilon^3) \tag{4.73}$$

$${T_{ab}}^{(\theta)} = \mathcal{O}(\epsilon^2) \tag{4.74}$$

$$R_{abcd} = \mathcal{O}(1) \tag{4.75}$$

$$\mathcal{L}_{\text{int}} = \mathcal{O}(\epsilon) \tag{4.76}$$

$$S = \mathcal{O}(\epsilon) \tag{4.77}$$

$$\epsilon C_{ab} = \mathcal{O}(\epsilon^2) \tag{4.78}$$

$$G_{ab} = -\frac{1}{2}\epsilon\left(2{R^{c\ \ d}_{\ a\ b}}^{(0)}{\bar{h}_{cd}}^{(1)} + \Box^{(0)}{\bar{h}_{ab}}^{(1)}\right) + \mathcal{O}(\epsilon^2) \tag{4.79}$$

So in the decoupling limit of $\epsilon \to 0$,

4.5.1 Zeroth Order

Just the Kerr solution with no scalar.

4.5.2 First Order

$$\Box^{(0)}\left(\epsilon\theta^{(1)}\right) = -\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right)^{(0)} \tag{4.80}$$

$$\Box^{(0)}\theta^{(1)} = -\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right)^{(0)} \tag{4.81}$$

and

$$m_p^2 G_{ab}^{(1)} + \epsilon\mathcal{L}_{\text{int}}^{(0)} g_{ab}^{(0)} + 2\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right)^{(0)} = 0 \tag{4.82}$$

$$m_p^2 G_{ab}^{(1)} = 0 \tag{4.83}$$

$$\left(2{R^{c\ \ d}_{\ a\ b}}^{(0)} + \delta^c_a\delta^d_b\Box^{(0)}\right){\bar{h}_{cd}}^{(1)} = 0 \tag{4.84}$$

where a solution is ${\bar{h}_{cd}}^{(1)} = 0$.

4.5.3 Second Order

Now at $\mathcal{O}(\epsilon^2)$ order, assuming ${\bar{h}_{cd}}^{(1)} = 0$,

$$m_p^2 G_{ab}^{(2)} + \epsilon\mathcal{L}_{\text{int}}^{(1)} g_{ab}^{(0)} + 2\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right)^{(1)} = \partial_a\left(\epsilon\theta^{(1)}\right)\partial_b\left(\epsilon\theta^{(1)}\right) - \frac{1}{2}g_{ab}^{(0)}\partial_c\left(\epsilon\theta^{(1)}\right)\partial^c\left(\epsilon\theta^{(1)}\right) \tag{4.85}$$

which reduces to

$$G^{(2)}_{ab} = \underbrace{m_p^{-2}\left[\underbrace{-\epsilon\mathcal{L}^{(1)}_{\text{int}}g^{(0)}_{ab} - 2\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right)^{(1)}}_{-\epsilon C^{(1)}_{ab}} + \underbrace{\epsilon^2\partial_a\theta^{(1)}\partial_b\theta^{(1)} - \frac{1}{2}\epsilon^2 g^{(0)}_{ab}\partial_c\theta^{(1)}\partial^c\theta^{(1)}}_{T^{(2)}_{ab}}\right]}_{S^{(2)}_{ab}} \tag{4.86}$$

$$\implies -\frac{1}{2(2!)}\left(2R^{c\ \ d\ (0)}_{\ a\ \ b} + \delta^c_a\delta^d_b\Box^{(0)}\right)\bar{h}_{cd}^{\ (2)} = S^{(2)}_{ab} \tag{4.87}$$

4.5.4 Third Order

We need to find θ to second order in ϵ:

$$\Box^{(0)}\left(\frac{1}{2}\epsilon^2\theta^{(2)}\right) = -\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right)^{(2)} \tag{4.88}$$

$$\Box^{(0)}\theta^{(2)} = -\frac{2}{\epsilon}\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right)^{(2)} \tag{4.89}$$

Then we have to $\mathcal{O}(\epsilon^3)$ order, assuming $\bar{h}_{cd}^{\ (1)} = 0$,

$$G^{(3)}_{ab} = \underbrace{m_p^{-2}\left[\underbrace{-\epsilon\mathcal{L}^{(2)}_{\text{int}}g^{(0)}_{ab} - 2\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right)^{(2)}}_{-\epsilon C^{(2)}_{ab}} + \underbrace{\frac{1}{2}\epsilon^2\left(\partial_a\theta^{(1)}\partial_b\theta^{(2)} + \partial_a\theta^{(2)}\partial_b\theta^{(1)} - g^{(0)}_{ab}\partial_c\theta^{(1)}\partial^c\theta^{(2)}\right)}_{T^{(3)}_{ab}}\right]}_{S^{(3)}_{ab}} \tag{4.90}$$

$$\implies -\frac{1}{2(3!)}\left(2R^{c\ \ d\ (0)}_{\ a\ \ b} + \delta^c_a\delta^d_b\Box^{(0)}\right)\bar{h}_{cd}^{\ (3)} = S^{(3)}_{ab} \tag{4.91}$$

4.5.5 Observation

We see as expected, the part of each order of G_{ab} acting on the solely the highest derivative of the metric is always an operator of the form $2R^{c\ \ d\ (0)}_{\ a\ \ b} + \delta^c_a\delta^d_b\Box^{(0)}$. This comes from the product of the perturbation expansion always has the same form for terms that have a single combinatorial contribution.

4.6 Inner Product Space of Perturbations

A natural first attempt at an inner product of p_{ab}, q_{cd} in the space of first order stationary, axisymmmetric perturbations of a background metric $g^{(0)}_{ab}$ is

$$\langle p, q\rangle \equiv \int p^{ab} q_{ab} \sqrt{g_{(0)}} d^4x \tag{4.92}$$

$$\langle p, q\rangle = \int dt\, d\phi \int p_{ab}\, g^{ac}_{(0)}\, g^{bd}_{(0)}\, q_{cd} \sqrt{g_{(0)}} d^2x \tag{4.93}$$

where raising and lowering is done by the background metric. Note that in equation (4.93) is only true for stationary, axisymmetric, metrics. The t and ϕ integrals are always the same for all p_{ab} and q_{cd}, so we can factor it out of all inner products.

4.6.1 Trace-reverse and the Inner Product

As a reminder, $\bar{\bar{p}}_{ab} = p_{ab}$, because $\left(p_{ab} - \frac{2}{d} g^{(0)}_{ab} p\right) - \frac{2}{d} g^{(0)}_{ab} \left(p - \frac{2}{d} g^{(0)}_{ab}\, g^{ab}_{(0)} p\right) = p_{ab}$ and that

$$\bar{p}^{ab}\bar{q}_{ab} = \left(p^{ab} - \frac{2}{d} g^{ab}_{(0)} p\right)\left(q_{ab} - \frac{2}{d} g^{(0)}_{ab}\, q\right) \tag{4.94}$$

$$= p^{ab}q_{ab} - \frac{2}{d}pq - \frac{2}{d}pq + \frac{4}{d}\cancel{\frac{g^{ab}_{(0)} g^{(0)}_{ab}}{d}} pq \tag{4.95}$$

$$= p^{ab}q_{ab} \tag{4.96}$$

$$\implies \langle p, q\rangle = \langle \bar{p}, \bar{q}\rangle \tag{4.97}$$

4.6.2 Self-Adjointness of the Linearized Einstein Operator

Reading off the form of the linearized Einstein operator $G^{(1)}$ in Lorenz gauge from eq. (4.84),

$$\langle p, G^{(1)}[q]\rangle = \int d^4x \sqrt{g_{(0)}}\, p^{ab} G^{(1)}[q]_{ab} \tag{4.98}$$

$$= \int d^4x \sqrt{g_{(0)}}\, p^{ab} \left(2R^{c\ d\ (0)}_{\ a\ b} + \delta^c_a \delta^d_b \Box^{(0)}\right) \bar{q}_{cd} \tag{4.99}$$

$$= \int d^4x \sqrt{g_{(0)}} \left(2R^{a\ b\ (0)}_{\ c\ d} p_{ab} \bar{q}^{cd} + p^{cd} \Box^{(0)} \bar{q}_{cd}\right) \tag{4.100}$$

$$= \int d^4x \sqrt{g_{(0)}} \left(\overline{2R^{a\ b\ (0)}_{\ c\ d} p_{ab}} q^{cd} + \bar{p}^{cd} \Box^{(0)} q_{cd}\right) \tag{4.101}$$

$$= \int d^4x \sqrt{g_{(0)}} \left(2R^{a\ b\ (0)}_{\ c\ d} \bar{p}_{ab} q^{cd} + \bar{p}^{cd} \Box^{(0)} q_{cd}\right) \tag{4.102}$$

where the last step is because we have a Ricci-flat background, so $R^{a\ b\ (0)}_{\ c\ d} g^{(0)}_{ab} = 0 = R^{a\ b\ (0)}_{\ c\ d} g^{cd}_{(0)}$. And in general, we see that the trace-reverse operator commutes with $G^(1)$, i.e. for all q, $\overline{G^{(1)}[\bar{q}]} = G^{(1)}[q]$.

Examining the second term of the integral, we integrate by parts twice and make use of the use the identity (A.18),

$$\int d^4x\sqrt{g_{(0)}}\,\bar{p}^{cd}\tilde{\nabla}_a\tilde{\nabla}^a q_{cd} = \int d^4x\sqrt{g_{(0)}}\,\tilde{\nabla}_a(\bar{p}^{cd}\tilde{\nabla}^a q_{cd}) - \int d^4x\sqrt{g_{(0)}}\,\tilde{\nabla}_a\bar{p}^{cd}\tilde{\nabla}^a q_{cd} \tag{4.103}$$

$$= \cancel{\int d^4x\,\partial_a(\sqrt{g_{(0)}}\bar{p}^{cd}\tilde{\nabla}^a q_{cd})} - \int d^4x\sqrt{g_{(0)}}\,\tilde{\nabla}_a\bar{p}^{cd}\tilde{\nabla}^a q_{cd} \tag{4.104}$$

$$= -\int d^4x\sqrt{g_{(0)}}\,\tilde{\nabla}^a(\tilde{\nabla}_a\bar{p}^{cd}q_{cd}) + \int d^4x\sqrt{g_{(0)}}\,\tilde{\nabla}^a\tilde{\nabla}_a\bar{p}^{cd}q_{cd} \tag{4.105}$$

$$= -\cancel{\int d^4x\,\partial_a(\sqrt{g_{(0)}}\tilde{\nabla}^a\bar{p}^{cd}q_{cd})} + \int d^4x\sqrt{g_{(0)}}\,\tilde{\nabla}^a\tilde{\nabla}_a\bar{p}^{cd}q_{cd} \tag{4.106}$$

Therefore, we have

$$\langle p, G^{(1)}[q]\rangle = \int d^4x\sqrt{g_{(0)}}\,\left(2R^{a\ b\ (0)}_{\ c\ d} + \delta^a_c\delta^b_d\Box^{(0)}\right)\bar{p}_{ab}q^{cd} \tag{4.107}$$

$$= \int d^4x\sqrt{g_{(0)}}\,G^{(1)}[p]^{cd}q_{cd} \tag{4.108}$$

$$= \langle G^{(1)}[p], q\rangle \tag{4.109}$$

The operator $G^{(1)}$ is self-adjoint with respect to this inner product.

4.7 Bianchi Identity

4.7.1 General Connections

Baez and Muniain[1] outline an elegant proof of the Bianchi identity, reproduced here in detail. We will use the the Bianchi identity to show the geometric origin of the divergencelessness of the Einstein tensor and all possible source terms.

Given a fiber bundle $\pi : E \to \mathcal{M}$ and a connection D on $\mathcal{M}$, for any E-valued form $\eta = s_I \otimes \omega^I$ on $\mathcal{M}$, in local coordinates,

$$\mathrm{d}_D^2\eta = \mathrm{d}_D\left(D_\nu s_I \otimes \mathrm{d}x^\nu \wedge \mathrm{d}x^I\right) \tag{4.110}$$

$$= D_\mu D_\nu s_I \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I \tag{4.111}$$

$$= \frac{1}{2}\left[D_\mu, D_\nu\right] s_I \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I \tag{4.112}$$

$$= \frac{1}{2}F_{\mu\nu}s_I \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I \tag{4.113}$$

$$= F \wedge \eta \tag{4.114}$$

Note that the exterior covariant derivative doesn't form a de Rham cohomology where $\mathrm{d}^2 = 0$ because the covariant derivative is not commutative, unlike the partial derivative. The *failure* to commute is the geometric curvature.

Therefore,

$$
\begin{aligned}
\mathrm{d}_D^3\eta &= \mathrm{d}_D(\mathrm{d}_D^2\eta) && (4.115)\\
&= \mathrm{d}_D(F \wedge \eta) && (4.116)\\
&= \mathrm{d}_D F \wedge \eta + F \wedge \mathrm{d}_D\eta && (4.117)\\
\mathrm{d}_D^3\eta &= \mathrm{d}_D^2(\mathrm{d}_D\eta) && (4.118)\\
&= F \wedge \mathrm{d}_D\eta && (4.119)\\
\implies & \boxed{\mathrm{d}_D F = 0} && (4.120)
\end{aligned}
$$

In local coordinates,

$$
\begin{aligned}
0 = \mathrm{d}_D F \wedge \eta = \mathrm{d}_D\left(\frac{1}{2}F_{\mu\nu} \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right) \wedge \left(s_I \otimes \mathrm{d}x^I\right) && (4.121)\\
= \frac{1}{2}(D_\lambda F_{\mu\nu}) \otimes \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \left(s_I \otimes \wedge \mathrm{d}x^I\right) && (4.122)\\
= \frac{1}{2}(D_\lambda F_{\mu\nu}) s_I \otimes \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (4.123)\\
= \frac{1}{2}\left(D_\lambda\left(F_{\mu\nu}s_I\right) - F_{\mu\nu}\left(D_\lambda s_I\right)\right) \otimes \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (4.124)\\
= \frac{1}{2}[D_\lambda, F_{\mu\nu}] s_I \otimes \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (4.125)\\
= \frac{1}{2}\cdot\frac{1}{3}\left([D_\lambda, F_{\mu\nu}] + [D_\mu, F_{\nu\lambda}] + [D_\nu, F_{\lambda\mu}]\right) s_I \otimes \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (4.126)\\
\implies 0 = [D_\lambda, F_{\mu\nu}] + [D_\mu, F_{\nu\lambda}] + [D_\nu, F_{\lambda\mu}] && (4.127)\\
\boxed{0 = [D_\lambda, [D_\mu, D_\nu]] + [D_\mu, [D_\nu, D_\lambda]] + [D_\nu, [D_\lambda, D_\mu]]} && (4.128)
\end{aligned}
$$

which is in the form of the Jacobi identity.

4.7.2 With Riemann curvature

For our Levi-Civita connection ∇ compatible with metric g, we have the curvature

$$
R(u,v)w = \left([\nabla_u, \nabla_v] - \nabla_{[u,v]}\right) w, \tag{4.129}
$$

which is just the curvature of the connection ∇.

$$
\begin{aligned}
0 &= [u,[v,w]] + [v,[w,u]] + [w,[u,v]] && (4.130)\\
&= \nabla_u[v,w] - \nabla_{[v,w]}u + (uvw \text{ cyc}) && (4.131)\\
&= \nabla_u(\nabla_v w - \nabla_w v) - \nabla_{[v,w]}u + (uvw \text{ cyc}) && (4.132)\\
&= [\nabla_u, \nabla_v]w - \nabla_{[u,v]}w + (uvw \text{ cyc}) && (4.133)\\
0 &= R(\nabla_u, \nabla_v)w + (uvw \text{ cyc}) && (4.134)\\
& && (4.135)
\end{aligned}
$$

Specifically, the Riemann curvature tensor is $R^a{}_{bcd} e_a \equiv R(\nabla_b, \nabla_c) e_d$. Choose $u = \partial_a, v = \partial_b, w = \partial_c$ to be coordinate basis vector fields.

$$\implies 0 = R(\nabla_a, \nabla_b)\partial_c + R(\nabla_b, \nabla_c)\partial_a + R(\nabla_c, \nabla_a)\partial_b \tag{4.136}$$

$$\implies 0 = R^d{}_{abc} + (abc \text{ cyc}) \tag{4.137}$$

$$\implies \boxed{0 = R^d{}_{[abc]}} \tag{4.138}$$

From eq. (4.128) applied to the Levi-Civita connection,

$$0 = [\nabla_a, [\nabla_b, \nabla_c]] + [\nabla_b, [\nabla_c, \nabla_a]] + [\nabla_c, [\nabla_a, \nabla_b]] \tag{4.139}$$

$$= [\nabla_a, R(\nabla_b, \nabla_c)] e_d + (abc \text{ cyc}) \tag{4.140}$$

$$= \nabla_a R^e{}_{bcd} e_e - \cancel{R(\nabla_b, \nabla_c) g_{ad}} + (abc \text{ cyc}) \tag{4.141}$$

$$\implies 0 = \nabla_a R_{ebcd} + (abc \text{ cyc}) \tag{4.142}$$

$$\implies 0 = \nabla_a R_{ecdb} + \nabla_a R_{edbc} + (abc \text{ cyc}) \tag{4.143}$$

where we use eq. (4.137) in the last step.

Contracting with the metric twice,

$$0 = g^{ec} \left(\cancel{\nabla_a R_{ecdb}} + \nabla_a R_{edbc} + (abc \text{ cyc}) \right) \tag{4.144}$$

$$0 = -\nabla_a R_{db} + \nabla_b R_{da} + \nabla^e R_{edab} \tag{4.145}$$

$$0 = g^{bd} \left(-\nabla_a R_{db} + \nabla_b R_{da} + \nabla^e R_{edab} \right) \tag{4.146}$$

$$0 = -\nabla_a R + \nabla^d R_{da} + \nabla^e R_{ea} \tag{4.147}$$

$$\implies 0 = \nabla^d \underbrace{(2R_{da} - g_{da} R)}_{2G_{da}} \tag{4.148}$$

$$\implies \boxed{0 = \nabla^d G_{da}} \tag{4.149}$$

5 Analytical Results

The following are metric dependent results and lead us into Weyl-Lewis-Papaterou black hole spacetime that we are considering for this project.

5.1 Birkhoff's Theorem

Here is a nice (full) proof of Birkhoff's theorem. The main idea comes from Eric Poisson.[2]

5.1.1 Spherical Symmetry

Assuming a spherically symmetric $3+1$ dimensional spacetime, we can choose coordinates so that the metric has the general form:

$$ds^2 = A(t,r)dt^2 + B(t,r)dt\,dr + C(t,r)dr^2 + D(t,r)d\Omega^2 \tag{5.1}$$

We can transform our coordinates (t,r) so that r becomes $\sqrt{D}$. We choose the positive root because we want the angular coordinates to have positive Lorentzian signature (If we choose the negative convention our final metric change to reflect the convention change). Therefore we can always rewrite our spherically symmetric metric as

$$ds^2 = A(t,r)dt^2 + B(t,r)dt\,dr + C(t,r)dr^2 + r^2 d\Omega^2 \tag{5.2}$$

where we have chosen the coordinate r specifically to give the spatial 2-sphere an r^2 areal dependence in the 4-fold.

Given any $A(t,r), B(t,r), C(t,r)$, we can transform the t coordinates so that our new coordinates, $t'(t,r)$ and r, gives

$$dt'^2 = \left(\frac{\partial t'}{\partial t}dt + \frac{\partial t'}{\partial r}dr\right)^2 \tag{5.3}$$

$$dt'^2 = \left(\frac{\partial t'}{\partial t}\right)^2 dt^2 + 2\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}dtdr + \left(\frac{\partial t'}{\partial r}\right)^2 dr^2 \tag{5.4}$$

$$D(t',r)\left(\frac{\partial t'}{\partial t}\right) = A(t,r) \tag{5.5}$$

$$D(t',r)\left(2\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}\right) = B(t,r) \tag{5.6}$$

$$E(t',r) - D(t',r)\left(\frac{\partial t'}{\partial r}\right)^2 = C(t,r) \tag{5.7}$$

Since we have three equations for three variables $t'(t,r), D\left(t'(t,r),r\right), E\left(t'(t,r),r\right)$, the equations are always soluble up given initial conditions. The choice of initial conditions is part of the gauge choice of our coordinate system. Then the line element is

$$ds^2 = D(t,r)dt^2 + E(t,r)dr^2 + r^2 d\Omega^2 \tag{5.8}$$

We see that we have two functional degrees of freedom assuming spherical symmetry. Once the vacuum Einstein Field Equations are imposed, we will see that only a real valued parameter will remain as a degree of freedom.

5.1.2 Vacuum Einstein Field Equations

In regions where D and E do not blow up or go to 0, we can renaming our metric degrees of freedom, in two steps:

$$ds^2 = -e^{2\psi(t,r)} f(t,r) dt^2 + \frac{1}{f(t,r)} dr^2 + r^2 d\Omega^2 \tag{5.9}$$

$$ds^2 = -e^{2\psi(t,r)} \left(1 - \frac{2m(t,r)}{r}\right) dt^2 + \left(1 - \frac{2m(t,r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{5.10}$$

In complete vacuum ${T^\mu}_\nu = 0$, we have that for the Einstein tensor ${G^\mu}_\nu$ with the help of Mathematica,

$$0 = {G^t}_t = \frac{-2\partial_r m(t,r)}{r^2} \tag{5.11}$$

$$0 = {G^r}_t = \frac{2\partial_t m(t,r)}{r^2} \tag{5.12}$$

$$0 = {G^r}_r - {G^t}_t = \frac{2}{r}\left(1 - \frac{2m(t,r)}{r}\right) \partial_r \psi(t,r) \tag{5.13}$$

By equation (5.11), $m(t,r) = m(t)$ and by equation (5.12), $m(t,r) = m(r)$. Therefore $m(t,r)$ is a real constant.
Now by equation (5.13), we have $\psi(t,r) = \psi(t)$.
We can then rescale $t \mapsto e^{-\psi(t)} t$, so that $g_{tt} = -\left(1 - \frac{2m}{r}\right)$ and all other metric components stay the same.

Therefore the unique spherically symmetric solution to the vacuum Einstein Field equations with $\Lambda = 0$ is the Schwarzschild solution:

$$\boxed{ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2} \tag{5.14}$$

for some coordinates with the $- + ++$ Lorentzian signature.

Notice we see that any spherically symmetric solution must be asymptotically flat (as $r \to \infty$) and static (with respect to the time-like vector $\frac{\partial}{\partial t}$); we did not impose these conditions.

Therefore, there is no gravitational monopole radiation.

5.1.3 Komar Mass

It turns out the Komar mass integral of the Schwarzschild solution is m, so m really does correspond to a physical mass of the metric.

5.2 Weyl-Lewis-Papapetrou

We would like to do something like Birkhoff's theorem and the Schwarzschild solution, but for axisymmetric spacetimes not spherically-symmetric spacetimes. Birkhoff's theorem does not exist for axisymmetric spacetimes, but we can find the unique stationary axisymmetric metric: Weyl-Lewis-Papapetrou (WLP). WLP is our gauge of choice for most of our analytical analysis. In order to prove its uniqueness we need a little machinery called Frobenius' Theorem.

5.2.1 Frobenius' Theorem

There are a few equivalent statements of Frobenius' Theorem; while the differential form version is nice, we use the vector field form for our current purposes. Frobenius' Theorem is useful not only for the proof of uniqueness of the WLP metric, but also will be used to show the integrability conditions for the solution to the Einstein Field Equations under a WLP metric.

Without introducing to many definitions, the theorem is roughly

Theorem 5.1 *In order to have a smooth sub-manifold of $\mathcal{M}$ that has tangent spaces coinciding with a tangent sub-bundle $W \subseteq E$ over $\mathcal{M}$, it is necessary and sufficient for W to be involute, i.e. $\forall X^a, Y^a \in W : [X, Y]^a \in W$.*

Therefore we have the following corollary:

Corollary 5.1.1 *If vector fields X^a and Y^a commute, with either vanishing at a point, and*

$$X^a R_a{}^{[b} X^c Y^{d]} = 0 = Y^a R_a{}^{[b} Y^c X^{d]}, \tag{5.15}$$

then the 2-fold orthogonal to X^a and Y^a are integrable.

The proofs are outlined in Wald[3], and may be reproduced here at a later time.

5.2.2 Proof of WLP

Given a time-like $\left(\frac{\partial}{\partial t}\right)^a$ and an "azimuthal" space-like $\left(\frac{\partial}{\partial \phi}\right)^a$ Killing vector fields for stationary axisymmetric $1+3$ dimensional spacetimes. Assuming these satisfy corollary 5.1.1, the span of the other vector fields generated by the other two coordinates (x_2 and x_3) are orthogonal to ∂_t^a and ∂_ϕ^a. (The first condition of corollary 5.1.1 is trivial, but for the second there is a possible argument based on t- and ϕ-reversal symmetry, but further investigation is needed.)

$$ds^2 = V(x_2, x_3)dt^2 + 2W(x_2, x_3)dtd\phi + X(x_2, x_3)d\phi^2 + g_{ij}(x_2, x_3)dx^i dx^j \tag{5.16}$$

for $i, j \in \{2, 3\}$. In block matrix form, the metric is

$$g_{ab} = \begin{pmatrix} -V & W & 0 & 0 \\ W & X & 0 & 0 \\ 0 & 0 & g_{22} & g_{23} \\ 0 & 0 & g_{23} & g_{33} \end{pmatrix} \tag{5.17}$$

Note that there are six distinct functions of x_2 and x_3.

We choose $x_2 = \rho = VX + W^2$, which is the negative of determinant of the upper 2×2 block. And choose $x_3 = z$ be such that $\nabla_a \rho \nabla^a z = 0$. Redefining variables, we must have

$$ds^2 = -V(dt - wd\phi)^2 + V^{-1}\rho^2 d\phi^2 + \Omega^2(d\rho^2 + \Lambda dz^2) \tag{5.18}$$

where $w = W/V$, $\Omega^2 = g_{22}$, and $\Lambda = g_{33}/\Omega^2$.

The four functional degrees of freedom are $V(\rho, z), w(\rho, z), \Omega(\rho, z), \Lambda(\rho, z)$.

We have made a gauge transformation to the unique Weyl-Lewis-Papapetrou coordinates for any stationary, axisymmetric spacetime, up to univariate scaling of z.

5.3 Schwarzschild in Weyl-Lewis-Papapetrou

5.3.1 Schwarzschild Background

We want to describe spacetimes in with a Schwarzschild background. Therefore we expect there to exist $V = V_0 + \delta V, w = w_0 + \delta w, \Omega = \Omega_0 + \delta\Omega, \Lambda = \Lambda_0 + \delta\Lambda$, where the variables with the naught-subscripts describe Schwarzschild background metric, and the δ variables are perturbations that keep the metric stationary and axisymmetric. Let's solve for the Schwarzschild solution only in terms of the background first, with no perturbations; we need to get the metric into the form:

$$ds^2 = -V_0(dt - w_0 d\phi)^2 + V_0^{-1}\rho^2 d\phi^2 + \Omega_0^2(d\rho^2 + \Lambda_0 dz^2) \tag{5.19}$$

Note that at the end of our calculation, we expect to choose coordinates so that $\Lambda_0 = 1$ because Schwarzschild is Ricci-flat.

5.3.2 Motivation of WLP Coordinates

By Birkhoff's Theorem, the Schwarzschild metric (5.14) is axisymmetric and stationary (in fact it is static):

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.20}$$

Therefore we should be able to write the metric in Weyl-Lewis-Papapetrou form.

We keep the time and azimuthal directions the same, as it is natural to pick $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ as our Killing vector fields. Therefore were are transforming the spatial coordinates r and θ only, from those that are spherically symmetric to those cylindrically symmetric.

We identify that $V_0 = 1 - \frac{2m}{r}$ and $w_0 = 0$, so our metric is in the form:

$$ds^2 = -V_0(dt - w_0 d\phi)^2 + V_0^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.21}$$

We see that the standard spherical to cylindrical ($r\sin\theta \mapsto \rho$, $r\cos\theta \mapsto z$) will not suffice because the only $d\phi^2$ term in the line element will be $r^2\sin^2\theta d\phi^2 \mapsto \rho^2 d\phi^2$, and in the WLP form, we need $V_0^{-1}\rho^2 d\phi^2$. Thus, we make our transformation $V_0^{1/2} r\sin\theta \mapsto \rho$, so that $r^2\sin^2\theta d\phi^2 \mapsto V_0^{-1}\rho^2 d\phi^2$.

Our transformation is so far defined by

$$\rho = V_0^{1/2} r\sin\theta = \sqrt{r^2 - 2mr}\sin\theta \tag{5.22}$$

$$\implies d\rho = \frac{r - m}{V_0^{1/2} r}\sin\theta dr + \underbrace{V_0^{1/2} r\cos\theta}_{\tilde{\rho}}\, d\theta \tag{5.23}$$

We see that $\tilde{\rho} = V_0^{1/2} r\cos\theta$ is the trigonometric conjugate of $\rho = V_0^{1/2} r\sin\theta$ (i.e. $\tilde{\rho}^2 + \rho^2 = V_0 r^2$). And with a clever definition of z, we have

$$z = (r - m)\cos\theta \tag{5.24}$$

$$\implies dz = \cos\theta dr - \underbrace{(r - m)\sin\theta}_{\tilde{z}}\, d\theta \tag{5.25}$$

where $\tilde{z} = (r - m)\sin\theta$ is the trignometric conjugate of $z = (r - m)\cos\theta$.

We see a good sign that $\frac{\tilde{z}}{V_0^{1/2} r}$ appears in (5.23) and $\frac{\tilde{\rho}}{V_0^{1/2} r}$ appears in (5.25).
So with this transformation:

$$\boxed{\begin{aligned} t &= t && (5.26)\\ \rho &= V_0^{1/2} r \sin\theta = \sqrt{r^2 - 2mr}\sin\theta && (5.27)\\ z &= (r-m)\cos\theta && (5.28)\\ \phi &= \phi && (5.29)\end{aligned}}$$

we have

$$\boxed{\begin{aligned} dt &= dt && (5.30)\\ d\rho &= V_0^{-1/2} r^{-1}\tilde{z}dr + \tilde{\rho}\mathrm{d}z && (5.31)\\ dz &= V_0^{-1/2} r^{-1}\tilde{\rho}dr - \tilde{z}d\theta && (5.32)\\ d\phi &= d\phi && (5.33)\end{aligned}}$$

Therefore, we have in terms of the auxiliary variables $\tilde{\rho} = V_0^{1/2} r\cos\theta$ and $\tilde{z} = (r-m)\sin\theta$,

$$\implies \tilde{z}d\rho + \tilde{\rho}dz = V_0^{-1/2} r^{-1}(\tilde{z}^2 + \tilde{\rho}^2)dr \tag{5.34}$$

$$\implies \boxed{dr = \frac{V_0^{1/2} r}{\tilde{z}^2 + \tilde{\rho}^2}(\tilde{z}d\rho + \tilde{\rho}dz)} \tag{5.35}$$

$$\implies \tilde{\rho}d\rho - \tilde{z}dz = (\tilde{z}^2 + \tilde{\rho}^2)d\theta \tag{5.36}$$

$$\implies \boxed{d\theta = \frac{1}{\tilde{z}^2 + \tilde{\rho}^2}(\tilde{\rho}d\rho - \tilde{z}dz)} \tag{5.37}$$

Substituting into the metric,

$$ds^2 = -V_0(dt - w_0 d\phi)^2 + V_0^{-1}\rho^2 d\phi^2 + \cancel{V_0^{-1}}\frac{\cancel{V_0}r^2}{(\tilde{z}^2 + \tilde{\rho}^2)^2}(\tilde{z}d\rho + \tilde{\rho}dz)^2 + r^2\frac{(\tilde{\rho}d\rho - \tilde{z}dz)^2}{(\tilde{z}^2 + \tilde{\rho}^2)^2} \tag{5.38}$$

$$ds^2 = -V_0(dt - w_0 d\phi)^2 + V_0^{-1}\rho^2 d\phi^2 + \frac{r^2}{(\tilde{z}^2 + \tilde{\rho}^2)^{\cancel{2}}}\left(\cancel{(\tilde{z}^2 + \tilde{\rho}^2)}d\rho^2 + \cancel{(\tilde{z}^2 + \tilde{\rho}^2)}dz^2\right) \tag{5.39}$$

$$ds^2 = -V_0(dt - w_0 d\phi)^2 + V_0^{-1}\rho^2 d\phi^2 + \frac{r^2}{\tilde{z}^2 + \tilde{\rho}^2}\left(d\rho^2 + dz^2\right) \tag{5.40}$$

We see that we've chosen z correctly so that $\Lambda_0 = 1$ and

$$\Omega_0^2 = \frac{r^2}{\tilde{z}^2 + \tilde{\rho}^2} = \frac{r^2}{(r^2 - 2mr + m^2)\sin^2\theta + (r^2 - 2mr)\cos^2\theta} \tag{5.41}$$

$$= \frac{r^2}{(r^2 - 2mr) + m^2\sin^2\theta} \tag{5.42}$$

Therefore we have for the Schwarzschild background

$$ds^2 = -V_0(dt - w_0 d\phi)^2 + V_0^{-1}\rho^2 d\phi^2 + \Omega_0^2\left(d\rho^2 + \Lambda_0 dz^2\right) \tag{5.43}$$

So our Weyl-Lewis-Papapetrou functional degrees of freedom are, as functions (r, θ),

$$\boxed{\begin{aligned} V &= \left(1 - \frac{2m}{r}\right) + \delta V && (5.44)\\ w &= 0 + \delta w && (5.45)\\ \Omega^2 &= \frac{r^2}{(r^2 - 2mr) + m^2 \sin^2\theta} + \delta\Omega^2 && (5.46)\\ \Lambda &= 1 + \delta\Lambda && (5.47) \end{aligned}}$$

5.3.3 Coordinate Singularities of Background Schwarzschild

Despite the curvature singularity at $r = 0$, we have coordinate singularities when $\Omega_0^2 \to \infty$, i.e.

$$0 = r^2 - 2mr + m^2 \sin^2\theta \tag{5.48}$$

$$0 = (r - m)^2 - m^2 \cos^2\theta \tag{5.49}$$

$$0 = (\underbrace{r - m + m\cos\theta}_{R_+})(\underbrace{r - m - m\cos\theta}_{R_-}) \tag{5.50}$$

With the auxiliary variables $R_\pm$, we rewrite our WLP functions with the substitution $r = \frac{1}{2}(R_+ + R_- + 2m)$:

$$\boxed{\begin{aligned} V &= \frac{R_+ + R_- - 2m}{R_+ + R_- + 2m} + \delta V && (5.51)\\ w &= 0 + \delta w && (5.52)\\ \Omega^2 &= \frac{(R_+ + R_- + 2m)^2}{4R_+R_-} + \delta\Omega^2 && (5.53)\\ \Lambda &= 1 + \delta\Lambda && (5.54) \end{aligned}}$$

$$\begin{aligned} \rho^2 + z^2 &= (r^2 - 2mr)\sin^2\theta + (r - m)^2\cos^2\theta && (5.55)\\ &= (r - m)^2 + m^2\cos^2\theta - m^2 && (5.56)\\ &= (r - m \pm m\cos\theta)^2 - m^2 \mp 2(r - m)m\cos\theta && (5.57)\\ &= R_\pm^2 - m^2 \mp 2mz && (5.58)\\ \implies \rho^2 + (z \pm m)^2 &= R_\pm^2 && (5.59)\\ \implies & \boxed{R_\pm = \sqrt{\rho^2 + (z \pm m)^2}} && (5.60) \end{aligned}$$

and thus our WLP functions are now functions of (ρ, z).

The coordinate singularities corresponding to $R_\pm = 0$ are now at $(\rho, z) = (0, \pm m)$ for all t and ϕ.

We also have a coordinate singularity when $\rho \to 0$, so all the coordinate singularities are at the line $\rho = 0$ in the spacetime, which includes the $(\rho, z) = (0, \pm m)$ singularity as well.

5.4 Mathematica for perturbations of Kerr and Schwarzschild

I was able to calculate the Einstein operator in WLP coordinates for both Kerr and Schwarzschild backgrounds. The Kerr solution in WLP form I used are from Jones and Wang[4]. The solutions with the explicit coordinates are too long to reproduce here in the progress report, but are included in the companion Mathematica files on the DCC.

There were many facts about WLP computed in Mathematica. One observation is that h_{ab} traceless in WLP. However WLP is confirmed to not be a Lorenz gauge. Furthermore, WLP does not seem to be obviously in a form of a generalized Harmonic gauge, but future analysis is needed.

5.5 The action of Einstein operator in WLP gauge: Ricci-flat

[This part was quite difficult, even for with the Ricci-flat simplification. The manipulations here are not referenced anywhere and it took a lot of sweat and trial and error to get the following result.]

5.5.1 Constraint equations

From $G_{ab} = 0$, we have ostensibly 6 non-zero equations of motion, which correspond to G_{00}, G_{03}, G_{33} and G_{11}, G_{12}, G_{22}.

From the first three, we can construct the combinations

$$e^{2\gamma}\left(\left(V^{-2}-\rho^{-2}w^2\right)G_{00}+e^{2\gamma}\rho^{-2}w^2\right)G_{33}=\vec{\nabla}\left(V^{-1}\vec{\nabla}V+\rho^{-2}V^2w\vec{\nabla}w\right) \tag{5.61}$$

$$e^{2\gamma}\rho^{-2}\left(wG_{00}+G_{03}\right)=\vec{\nabla}\left(\rho^{-2}V^2\vec{\nabla}w\right) \tag{5.62}$$

where $\vec{\nabla}$ is the gradient under the flat metric $ds^2 = \rho^2 d\phi^2 + d\rho^2 + dz^2$, *not* $ds^2 = g_{ab}dx^a dx^b$

We have $G_{00} = G_{03} = G_{33} = 0$ if and only if

$$0=\vec{\nabla}\cdot\left(V^{-1}\vec{\nabla}V+\rho^{-2}V^2w\vec{\nabla}w\right) \tag{5.63}$$

$$0=\vec{\nabla}\cdot\left(\rho^{-2}V^2\vec{\nabla}w\right) \tag{5.64}$$

and the Bianchi identity $\nabla^a G_{ab} = 0$ is satisfied.

Furthermore, we have $G_{11} = -G_{22}$ automatically, so we are left with

$$0=-G_{11}=G_{22}=\frac{1}{4V^2}\left((\partial_\rho V)^2-(\partial_z V)^2\right)-\frac{V^2}{4\rho^2}\left((\partial_\rho w)^2-(\partial_z w)^2\right)-\frac{\partial_\rho\gamma}{\rho^2} \tag{5.65}$$

$$0=G_{12}=\frac{\partial_z\gamma}{\rho^2}-\frac{1}{2V^2}(\partial_\rho V)(\partial_z V)+\frac{V^2}{2\rho^2}(\partial_\rho w)(\partial_z w) \tag{5.66}$$

which are compatible because given eqs. (5.63) and (5.64), $\partial_\rho\partial_z\gamma = \partial_z\partial_\rho\gamma$ is true.

We have shown that there are 4 equations (2 of which are compatible) consistent with 3 metric variables in the Ricci-flat case.

5.6 Non-Ricci-flat Perturbations of Ricci-flat Background

Since we know $G_{ab} = T^{(0)}_{ab} + \epsilon T^{(1)}_{ab} + \mathcal{O}(\epsilon^2)$ and $T^{(0)}_{ab} = 0$, for sake of brevity, we use the notation $T_{ab} \equiv T^{(1)}_{ab}$, so that for the order ϵ^1 term, $G^{(1)}_{ab} = T_{ab}$.

5.7 Linearized Einstein Field Equations of WLP perturbations

5.8 z Gauge Fixing

5.8.1 $z \mapsto f(z)$

We have a remaining gauge freedom in WLP, $z \mapsto f(z)$ keeps the metric in WLP form. We need to fix the gauge completely to perform explicit numerical calculations. The map $z \mapsto f(z)$ changes the WLP metric by

$$ds^2 = -V(dt - wd\phi)^2 + V^{-1}\left(\rho^2 d\phi^2 + e^{2\gamma}(d\rho^2 + e^{2\lambda}dz^2)\right) \tag{5.67}$$

$$\mapsto -V(dt - wd\phi)^2 + V^{-1}\left(\rho^2 d\phi^2 + e^{2\gamma}(d\rho^2 + e^{2\lambda}\left(\partial_z f\right)^2 dz^2)\right) \tag{5.68}$$

$$= -V(dt - wd\phi)^2 + V^{-1}\left(\rho^2 d\phi^2 + e^{2\gamma}(d\rho^2 + e^{2(\lambda + \log \partial_z f)}dz^2)\right) \tag{5.69}$$

So the gauge freedom is

$$\lambda \mapsto \lambda + \log \partial_z f \tag{5.70}$$

$$\implies \lambda_0 + \epsilon\delta\lambda \mapsto \lambda_0 + \epsilon\delta\lambda + \log \partial_z f \tag{5.71}$$

We first fix our gauge so that $\lambda_0 = 0$, so the remaining gauge freedom is, for any function $G(z)$ that is $\mathcal{O}(\epsilon)$,

$$\delta\lambda \mapsto \delta\lambda + \log \partial_z f \tag{5.72}$$

$$\partial_z\delta\lambda \mapsto \partial_z\delta\lambda + \underbrace{\frac{\partial_z^2 f}{\partial_z f}}_{G(z)} \tag{5.73}$$

which means once we fix our gauge with $G(z)$ completely we have the condition that

$$\partial_z\delta\lambda + G(z) = H(\rho, z) \tag{5.74}$$

for an *a priori* unknown function $H(\rho, z)$

From the six original linearized EFEs, and imposing the background Wald equations we have

$$\partial_\rho\delta\lambda = \rho(T_{11} - T_{22}) \tag{5.75}$$

$$\implies \partial_z\partial_\rho\delta\lambda = \rho\partial_z(T_{11} - T_{22}) \tag{5.76}$$

Taking the ρ partial derivative of eq. (5.74) yields,

$$\partial_\rho\partial_z\delta\lambda = \partial_\rho H \tag{5.77}$$

$$\implies \partial_\rho H = \rho\partial_z(T_{11} - T_{22}) \tag{5.78}$$

Assuming $H(\rho = R, z) = 0$, for some R (which could be ∞, we have

$$\implies H(\rho, z) = \int_R^\rho \rho' \partial_z\left(T_{11}(\rho', z) - T_{22}(\rho', z)\right) d\rho' + C(z) \tag{5.79}$$

for some arbitrary constant $C(z)$.

But this $C(z)$ degree of ambiguity for $H(\rho, z)$ is exactly the gauge degree of freedom $G(z)$ in eq. (5.74)! Therefore, we have

$$\partial_z \delta\lambda = \int_R^\rho \rho' \partial_z \left(T_{11}(\rho', z) - T_{22}(\rho', z) \right) d\rho' + \tilde{C}(z) \tag{5.80}$$

where $\tilde{C}(z) = C(z) - G(z)$.

For our numerical purposes, we can just set $\tilde{C}(z) = 0$ to completely fix our z gauge degree of freedom.

5.8.2 Flat Laplacian of $\delta\lambda$

Therefore we have explicitly, $\partial_\rho \delta\lambda$ and $\partial_z \delta\lambda$, so we can construct the flat laplacian of $\delta\lambda$ under the metric $ds^2 = \rho d\phi^2 + d\rho^2 + dz^2$,

$$\nabla^2 \delta\lambda = \left(\partial_\rho^2 + \frac{\partial_\rho}{\rho} + \partial_z^2 \right) \delta\lambda = \int_\infty^\rho \rho' \partial_z^2 \left(T_{11}(\rho', z) + T_{22}(\rho', z) \right) \, d\rho' + \rho \partial_\rho \left(T_{11} + T_{22} \right) + 2(T_{11} + T_{22}) \tag{5.81}$$

along with the flat laplacians of $\delta V, \delta w$, and $\delta\gamma$ we found earlier.

5.9 Dynamical Chern-Simons

5.9.1 Equations of Motion

With θ coupled to the Pontryagin density, the equations of motion are

$$G_{ab} + \epsilon C_{ab} = \partial_a \theta \partial_b \theta - \frac{1}{2} g_{ab} \partial^c \theta \partial_c \theta \tag{5.82}$$

$$\nabla^a \nabla_a \theta = -\frac{1}{16} \epsilon {}^* RR = -\frac{1}{16} \epsilon_{cdef} R_{ab}{}^{ef} R^{abcd} \tag{5.83}$$

where using Mathematica and xTensor,

$$C_{ab} = \frac{1}{8} \left[-\theta \epsilon_{bdef} \nabla_c \nabla^f R_{acde} - \nabla^c \theta \epsilon_{bcef} \nabla_d R_{adef} - \nabla^d \nabla^c \theta R_{acef} \epsilon_{bdef} - \nabla^c \theta \epsilon_{bdef} \nabla^f R_{acde} \right] + (a \leftrightarrow b) \tag{5.84}$$

5.9.2 Over a Kerr background

With the Kerr solution as the background, i.e. $g_{ab}^{(0)}$, we have

$$\Box^{(0)} \theta = -\epsilon {}^* R^{(0)} R^{(0)} \tag{5.85}$$

$$= 96 (GM)^2 \frac{a \mu r (3r^2 - a^2 \mu^2)(r^2 - 3a^2 \mu^2)}{\Sigma^6} \tag{5.86}$$

where the second line is from [5], in rationalized Boyer-Lindquist coordinates.

6 Numerical Results/Setup

6.1 Linearized equations

Despite the redundancies in $G_{ab} = T^{\text{eff}}_{ab}$, from section 5.8.2, we can now cast the linearized equations into the form

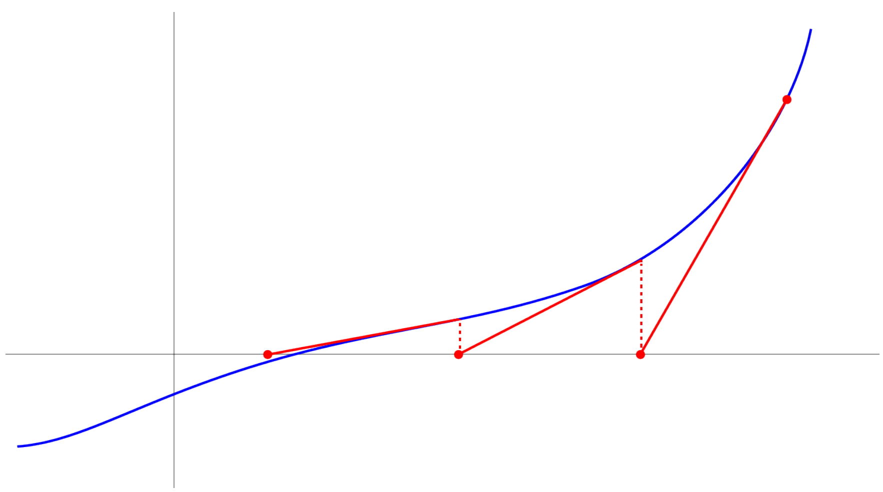
$$\Delta \begin{pmatrix} \delta V \\ \delta w \\ \delta \gamma \\ \delta \lambda \end{pmatrix} \equiv \Delta_0 \begin{pmatrix} \delta V \\ \delta w \\ \delta \gamma \\ \delta \lambda \end{pmatrix} + \text{lower order terms} = \text{source} \tag{6.1}$$

where Δ_0 is the induced 3-Laplacian of the Kerr background. The problem is now manifestly elliptic and well-posed. We can invert Δ_0 numerically, so in principle an iterative scheme can invert Δ.

6.2 Newton-Raphson method

Given this form, we use an iterative scheme to solve for these four metric variables in $\Delta \vec{v} = \vec{S}$

$$\begin{aligned} \text{initial guess: } \Delta_0 \vec{v}_0 &= \vec{S}_0 \\ \Delta(\vec{v}_0 + \delta \vec{v}) &= \vec{S} \\ \Delta \vec{v}_0 + \Delta_0 \delta \vec{v} &\approx \vec{S} \\ \implies \Delta_0 \delta \vec{v} &\approx \vec{S} - \Delta \vec{v}_0 \\ \text{iteratively solve: } &\boxed{\delta \vec{v} \approx \Delta_0^{-1} \left(\vec{S} - \Delta \vec{v}_0 \right)} \end{aligned}$$



This scheme is a generalization of Newton's method for root finding that we all know and love from elementary calculus.

6.3 Maximum Principle

We use the maximum principle to check a sign in our iterative scheme, to make sure it has a chance of converging,

Theorem 6.1 *Given a Laplacian D^2, and the differential equation of u of the form $D^2u = fu$ for some function $f \leq 0$ at all points, u cannot have a maximum in the interior of the domain.*

If u has a maximum at $\vec{x}_*$ on the interior of the domain then $D^2u(\vec{x}_*) < 0$ at some point the open neighborhood around $\vec{x}_*$. But then $fu \geq 0$ at that point in the neighborhood around $\vec{x}_*$. Contradiction.

Therefore, we need to make sure that f is not non-negative at all points in order to maximize u on the domain. In fact, in our application of this theorem $f < 0$ for all points.

6.4 Transformation to Rational-Polynomial Boyer-Lindquist Coordinates

Because we want to invert using the numerical scheme, and since Δ_0 is invertible in Boyer-Lindquist analytically, we convert our differential equations into Boyer-Lindquist to complete the Newton-Raphson method. We can show that for a $\lambda_0 = 0$, a Ricci-flat background, the background scalar laplacian is

$$\nabla^2_{\mathrm{WLP}} f = V_0 e^{-2\gamma_0} \left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2 \right) f(\rho, z) = \frac{1}{\Sigma} \left(\partial_r \Delta \partial_r + \partial_\mu (1-\mu^2) \partial_\mu \right) f(r, \mu) = \nabla^2_{\mathrm{BL}} f \quad (6.2)$$

So the principle part of the differential equations will remain exactly the same, with no extraneous lower order terms.

6.5 Loss of Gauge after Boyer-Lindquist Transformation

A preliminary result is the loss of the z gauge freedom once we transform into the Boyer-Lindquist coordinates.

6.6 Demonstration Model

We use non-minimally coupled scalar to the Pontryagin-Chern density, ${}^*RR \equiv -\frac{1}{2}\epsilon^{abcd} R_{abef} R_{cd}{}^{ef}$, over a Kerr background.

From eq. (4.81), we have the equation (with the conventional coupling factor of $\frac{1}{8}$ from [5])

$$\Box^{(0)}\theta^{(1)} = -\frac{1}{16}\epsilon^{abcd} R^{(0)}_{abef} R^{(0)}_{cd}{}^{ef} \quad (6.3)$$

7 Challenges

- Check if ADM equations are automatically satisfied or need to be constrained during the numerical evolution.
- Invert Linearized equation
- Relaxation code and numerics
- boundary conditions, and compactifying coordinates to bring in infinity.

8 Summary

We have made much progress so far this summer in analyzing black holes of alternative gravity theories. The analytics is mostly complete. However there are a few unresolved stumbling blocks with the numerics that prevent us from fully simulating these black holes.

The current goals are to finish simulating these black holes in dynamical Chern-Simons. Then apply the method to Einstein-dilaton-Gauss-Bonnet gravity. If a possible analytic inversion of Δ can be found then our method would drastically be simplified.

After this, we hope to implement the formalism directly in the Spectral Einstein Code (SpEC), and compute physically interesting quantities of these black holes, e.g. the thermodynamic entropy, the innermost stable circular orbit (ISCO), orbital frequencies, and the locations of the new horizons.

In the long term, we hope to work on non-stationary perturbations, looking at the quasi-normal modes of these black holes, computing solutions for binary black holes and their coalescences with these corrections to GR. Hopefully with these, we can generate gravitational waveforms at infinity and inform search pipelines of the gravitational wave detectors.

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Appendices

A Miscellaneous Identities Used in Proofs

A.0.1 Metric

$$
\begin{aligned}
\frac{\partial g^{ab}}{\partial g_{cd}} &= \frac{\partial (g^{aa'} g^{bb'} g_{a'b'})}{\partial g_{cd}} = \frac{\partial g^{aa'}}{\partial g_{cd}} g^{bb'} g_{a'b'} + g^{aa'} \frac{\partial g^{bb'}}{\partial g_{cd}} g_{a'b'} + g^{aa'} g^{bb'} \frac{\partial g_{a'b'}}{\partial g_{cd}} \\
&= \frac{\partial g^{aa'}}{\partial g_{cd}} \delta^{b}_{a'} + \frac{\partial g^{bb'}}{\partial g_{cd}} \delta^{a}_{b'} + g^{aa'} g^{bb'} \delta^{c}_{a'} \delta^{d}_{b'} \\
&= \frac{\partial g^{ab}}{\partial g_{cd}} + \frac{\partial g^{ba}}{\partial g_{cd}} + g^{ac} g^{bd} \\
\implies & \boxed{\frac{\partial g^{ab}}{\partial g_{cd}} = -g^{ac} g^{bd}}
\end{aligned}
\tag{A.1}
$$

A.0.2 Jacobi Formula

For a generic derivative operator ∂, one can show the following two facts:

$$
\log \det A = \operatorname{tr} \log A \tag{A.2}
$$

$$
\partial \operatorname{tr} F(A) = \operatorname{tr} \left(\frac{d}{dA} F(A) \partial A \right) \tag{A.3}
$$

Then one can prove:

$$
\begin{aligned}
\frac{1}{\det A} \partial \det A = \partial \log \det A &= \partial \operatorname{tr} \log A \\
&= \operatorname{tr} \left(\frac{d}{dA} \log A \partial A \right) \\
&= \operatorname{tr} \left(A^{-1} \partial A \right) \\
\implies & \boxed{\partial \det A = \det A \operatorname{tr} \left(A^{-1} \partial A \right)} \\
&= - \det A \operatorname{tr} \left(A (-A^{-2}) \partial A \right) \\
& \boxed{\partial \det A = - \det A \operatorname{tr} \left(A \partial (A^{-1}) \right)}
\end{aligned}
\tag{A.4}
$$

A.0.3 Metric Density

Let $g \equiv \det[g_{\mu\nu}]$ in this context. We use our result from (A.4). For variational derivatives w.r.t. to the inverse metric,

$$
\begin{aligned}
& \boxed{\delta g = -g g_{\mu\nu} \delta g^{\mu\nu}} \\
\delta \sqrt{-g} &= \frac{1}{2\sqrt{-g}} \times (-\delta g) \\
& \boxed{\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}}
\end{aligned}
\tag{A.5}
$$

For partial derivatives,

$$\boxed{\partial_i g = g g^{ab} \partial_i g_{ba}}$$
$$\partial_i \sqrt{-g} = \frac{1}{2\sqrt{-g}} \partial_i (-g) \tag{A.6}$$
$$\boxed{\partial_i \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{ab} \partial_i g_{ab}}$$

A.0.4 Connection Coefficients

$$\Gamma^i_{ij} = \frac{1}{2} g^{ia} (\cancel{\partial_i g_{aj}} + \partial_j g_{ia} - \cancel{\partial_a g_{ij}}) \tag{A.7}$$
$$= \frac{1}{2} g^{ia} \partial_j g_{ia} \tag{A.8}$$
$$\boxed{\Gamma^i_{ij} = \frac{1}{2g} \partial_j g} \tag{A.9}$$
$$\text{or } \boxed{\Gamma^i_{ij} = \frac{1}{\sqrt{-g}} \partial_j \sqrt{-g}} \tag{A.10}$$

$$g^{jk} \Gamma^i_{jk} = \frac{1}{2} g^{jk} g^{ia} (\partial_j g_{ak} + \partial_k g_{ja} - \partial_a g_{jk}) \tag{A.11}$$
$$= g^{jk} g^{ia} \partial_j g_{ka} - \frac{1}{2} g^{jk} g^{ia} \partial_a g_{jk} \tag{A.12}$$
$$= \cancel{g^{jk} \partial_j (g^{ia} g_{ka})} - g^{jk} \partial_j g^{ia} g_{ka} - \frac{1}{2} g^{ia} g^{jk} \partial_a g_{jk} \tag{A.13}$$
$$= -\frac{1}{\sqrt{-g}} \sqrt{-g} \partial_a g^{ia} - \frac{1}{\sqrt{-g}} g^{ia} \partial_a \sqrt{-g} \tag{A.14}$$
$$\boxed{g^{jk} \Gamma^i_{jk} = -\frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g} g^{ia} \right)} \tag{A.15}$$

A.0.5 Covariant Derivatives

$$\nabla_i v^i = \partial_i v^i + \Gamma^i_{ij} v^j \tag{A.16}$$
$$= \frac{1}{\sqrt{-g}} \sqrt{-g} \partial_i v^i + \frac{1}{\sqrt{-g}} \partial_j \sqrt{-g} v^j \tag{A.17}$$
$$\boxed{\nabla_i v^i = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} v^i)} \tag{A.18}$$
$$\Longrightarrow \boxed{\nabla_i \nabla^i \phi = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} \partial^i \phi)} \tag{A.19}$$

As a consistency check, we do the divergence of a covector field:

$$g^{ij}\nabla_i\omega_j = g^{ij}\partial_i\omega_j - g^{ij}\Gamma^k_{ij}\omega_k \tag{A.20}$$

$$= \frac{1}{\sqrt{-g}}\sqrt{-g}g^{ij}\partial_i\omega_j - \frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ik})\omega_k \tag{A.21}$$

$$\boxed{\nabla^i\omega_i = \frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\omega_j)} \tag{A.22}$$

which agrees with (A.18)

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