Notes

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Contents

1 Perturbation Theory

We have a 1-parameter family of geometries, described by

$$g_{ab}(\lambda) = {g_{ab}}^{(0)} + \lambda \left.\frac{dg_{ab}}{d\lambda}\right|_{\lambda=0} + \frac{\lambda^2}{2} \left.\frac{d^2 g_{ab}}{d\lambda^2}\right|_{\lambda=0} + \mathcal{O}(\lambda^3) \tag{1.1}$$

$$\equiv {g_{ab}}^{(0)} + \lambda h^{(1)}_{ab} + \frac{\lambda^2}{2} h^{(2)}_{ab} + \mathcal{O}(\lambda^3) \tag{1.2}$$

For our purposes at the moment, ${g_{ab}}^{(0)}$ is the background Ricci-flat spacetime (corresponding to Schwarzschild or Kerr), and $h^{(1)}_{ab} = \left.\frac{dg_{ab}}{d\lambda}\right|_{\lambda=0}$ is the first order metric perturbation.

2 Connection on a Background

We have the difference of connections, where $\nabla^{(\lambda)}_a$ is compatible with the metric ${g_{bc}}^{(\lambda)}$:

$$(\nabla^{(\lambda)}_a - \nabla^{(0)}_a) v^b = C^b_{ac} v^c \tag{2.1}$$

$$(\nabla^{(\lambda)}_a - \nabla^{(0)}_a) \omega_b = -C^c_{ab} \omega_c \tag{2.2}$$

where C^c_{ab} is a function of λ.

Therefore, from $0 = \nabla_c^{(\lambda)} {g_{ab}}^{(\lambda)}$, we have two identities:

$$C^c_{ab} = \frac{1}{2} g^{cd}{}_{(\lambda)} \left({\nabla_a}^{(0)} {g_{db}}^{(\lambda)} + {\nabla_b}^{(0)} {g_{ad}}^{(\lambda)} - {\nabla_d}^{(0)} {g_{ab}}^{(\lambda)} \right) \tag{2.3}$$

$$C^c_{ab} = \frac{1}{2} g^{cd}{}_{(\lambda)} \left(\partial_a {g_{db}}^{(\lambda)} + \partial_b {g_{ad}}^{(\lambda)} - \partial_d {g_{ab}}^{(\lambda)} \right) - \frac{1}{2} g^{cd}{}_{(0)} \left(\partial_a {g_{db}}^{(0)} + \partial_b {g_{ad}}^{(0)} - \partial_d {g_{ab}}^{(0)} \right) \tag{2.4}$$

For notational convenience let $\tilde{\nabla}_a \equiv {\nabla_a}^{(0)}$ and $\nabla_a \equiv {\nabla_a}^{(\lambda)}$. The Riemann curvature tensor is

$$\begin{aligned}
R_{abc}{}^d \omega_d &\equiv [\nabla_a, \nabla_b] \omega_c && (2.5)\\
&= \nabla_a \nabla_b \omega_c - (a \leftrightarrow b) && (2.6)\\
&= \tilde{\nabla}_a (\nabla_b \omega_c) - \cancel{C^d_{ab} (\nabla_d \omega_c)} - C^d_{ac} (\nabla_b \omega_d) - (a \leftrightarrow b) && (2.7)\\
&= \tilde{\nabla}_a (\tilde{\nabla}_b \omega_c - C^d_{bc} \omega_d) - C^d_{ac} (\tilde{\nabla}_b \omega_d - C^e_{bd} \omega_e) - (a \leftrightarrow b) && (2.8)\\
&= \tilde{\nabla}_a \tilde{\nabla}_b \omega_c - \tilde{\nabla}_a C^d_{bc} \omega_d \cancel{- C^d_{bc} \tilde{\nabla}_a \omega_d - C^d_{ac} \tilde{\nabla}_b \omega_d} + C^d_{ac} C^e_{bd} \omega_e - (a \leftrightarrow b) && (2.9)\\
&= \tilde{\nabla}_{[a} \tilde{\nabla}_{b]} \omega_c - \tilde{\nabla}_{[a} C^d_{b]c} \omega_d + C^d_{c[a} C^e_{b]d} \omega_e && (2.10)\\
&= \left(R^d{}_{abc}{}^{(0)} - \tilde{\nabla}_{[a} C^d_{b]c} + C^e_{c[a} C^d_{b]e} \right) \omega_d && (2.11)
\end{aligned}$$

$$\Longrightarrow \boxed{R_{abc}{}^d = R_{abc}{}^d{}_{(0)} - \tilde{\nabla}_{[a} C^d_{b]c} + C^e_{c[a} C^d_{b]e}} \tag{2.12}$$

3 Linearized Einstein Operator (possibly Lichnerowicz)

Let $\tilde{\nabla}_a \equiv {\nabla_a}^{(0)}$ and $g_{ab} = {g_{ab}}^{(\lambda)}$ unless otherwise specified.

$$\begin{aligned}
C^c_{ab} &= \frac{1}{2} g^{cd} \left(\tilde{\nabla}_a g_{db} + \tilde{\nabla}_b g_{ad} - \tilde{\nabla}_d g_{ab} \right) && (3.1)\\
C^c{}_{ab}{}^{(0)} &= 0 && (3.2)\\
\Longrightarrow C^c_{ab} &= \mathcal{O}(\lambda) && (3.3)\\
C^c_{ab} &= \frac{1}{2} \lambda g^{cd(0)} \left(\tilde{\nabla}_a h_{db} + \tilde{\nabla}_b h_{ad} - \tilde{\nabla}_d h_{ab} \right) + \mathcal{O}(\lambda^2) && (3.4)
\end{aligned}$$

We have

$$\begin{aligned}
R_{abc}{}^d &= R_{abc}{}^d{}_{(0)} - \tilde{\nabla}_{[a} C^d_{b]c} + \mathcal{O}(\lambda^2) && (3.5)\\
\Longrightarrow R_{ac} &= {R_{ac}}^{(0)} - \tilde{\nabla}_{[a} C^d_{d]c} + \mathcal{O}(\lambda^2) && (3.6)\\
&= {R_{ac}}^{(0)} - \frac{1}{2} \lambda g^{de(0)} \left(\tilde{\nabla}_a \tilde{\nabla}_d h_{ec} + \tilde{\nabla}_a \tilde{\nabla}_c h_{de} - \tilde{\nabla}_a \tilde{\nabla}_e h_{dc} - (a \leftrightarrow d) \right) + \mathcal{O}(\lambda^2) && (3.7)\\
&= {R_{ac}}^{(0)} - \frac{1}{2} \lambda \left([\tilde{\nabla}_a, \tilde{\nabla}^e] h_{ec} + \tilde{\nabla}_a \tilde{\nabla}_c h - \tilde{\nabla}^e \tilde{\nabla}_c h_{ae} - \tilde{\nabla}_a \tilde{\nabla}^d h_{dc} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ac} \right) + \mathcal{O}(\lambda^2) && (3.8)
\end{aligned}$$

$$\boxed{R_{ac} = {R_{ac}}^{(0)} - \frac{1}{2} \lambda \left(\tilde{\nabla}_a \tilde{\nabla}_c h - \tilde{\nabla}^e \tilde{\nabla}_{(a} h_{c)e} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ac} \right) + \mathcal{O}(\lambda^2)} \tag{3.9}$$

where we have $\tilde{\nabla}_a$ and h_{ab} raised and lowered (and traced) by the background metric $g^{cd}{}_{(0)}$.

Furthermore, we have

$$\begin{aligned}
R &= g^{ac} R_{ac} && (3.10)\\
&= (g^{ac(0)} - \lambda h^{ac}) {R_{ac}}^{(0)} - \frac{1}{2} \lambda g^{ac(0)} \left(\tilde{\nabla}_a \tilde{\nabla}_c h - \tilde{\nabla}^e \tilde{\nabla}_{(a} h_{c)e} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ac} \right) + \mathcal{O}(\lambda^2) && (3.11)\\
&= R^{(0)} - \lambda h^{ac} {R_{ac}}^{(0)} - \frac{1}{2} \lambda \left(\tilde{\nabla}_a \tilde{\nabla}^a h - 2 \tilde{\nabla}^e \tilde{\nabla}^a h_{ae} + \tilde{\nabla}_d \tilde{\nabla}^d h \right) + \mathcal{O}(\lambda^2) && (3.12)
\end{aligned}$$

$$\boxed{R = R^{(0)} - \lambda \left(h^{ac} {R_{ac}}^{(0)} + \tilde{\nabla}_d \tilde{\nabla}^d h - \tilde{\nabla}^c \tilde{\nabla}^d h_{cd} \right) + \mathcal{O}(\lambda^2)} \tag{3.13}$$

Therefore the linearized Einstein tensor is

$$
\begin{aligned}
G_{ab} \equiv & R_{ab} - \frac{1}{2} R g_{ab} & (3.14)\\
= & R_{ab}{}^{(0)} - \frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_{(a} h_{b)e} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ab} \right) & (3.15)\\
& - \frac{1}{2}(g_{ab}{}^{(0)} + \lambda h_{ab}) \left[R^{(0)} - \lambda \left(h^{cd} R_{cd}{}^{(0)} + \tilde{\nabla}_d \tilde{\nabla}^d h - \tilde{\nabla}^c \tilde{\nabla}^d h_{cd} \right) \right] + \mathcal{O}(\lambda^2) & (3.16)\\
= & G_{ab}{}^{(0)} - \frac{1}{2}\lambda h_{ab} R^{(0)} + \frac{1}{2}\lambda g_{ab}{}^{(0)} h^{cd} R_{cd}{}^{(0)} - \frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_{(a} h_{b)e} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ab} \right) & (3.17)\\
& + \frac{1}{2}\lambda g_{ab}{}^{(0)} \left(h^{cd} R_{cd}{}^{(0)} + \tilde{\nabla}_d \tilde{\nabla}^d h - \tilde{\nabla}^c \tilde{\nabla}^d h_{cd} \right) + \mathcal{O}(\lambda^2) & (3.18)
\end{aligned}
$$

If we have a Ricci-flat background, $R_{cd}{}^{(0)} = 0$,

$$
\boxed{G_{ab} = -\frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_{(a} h_{b)e} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ab} - g_{ab}{}^{(0)} \tilde{\nabla}_d \tilde{\nabla}^d h + g_{ab}{}^{(0)} \tilde{\nabla}^c \tilde{\nabla}^d h_{cd} \right) + \mathcal{O}(\lambda^2)} \qquad (3.19)
$$

which agrees with the Fierz-Pauli equation for massless spin-2 bosons in a Minkowski background.

We can also note that $\lambda \nabla_a = \lambda \tilde{\nabla}_a + \mathcal{O}(\lambda^2)$, so

$$
G_{ab} = -\frac{1}{2}\lambda \left(\nabla_a \nabla_b h - \nabla^e \nabla_{(a} h_{b)e} + \nabla_d \nabla^d h_{ab} - g_{ab} \nabla_d \nabla^d h + g_{ab} \nabla^c \nabla^d h_{cd} \right) + \mathcal{O}(\lambda^2) \qquad (3.20)
$$

4 Gauge conditions

4.1 Covariant Derivative Commutator derivation

Given that $[\tilde{\nabla}_a, \tilde{\nabla}_b]\omega_c = -R^d{}_{cab}{}^{(0)} \omega_d$, we have

$$
\begin{aligned}
[\tilde{\nabla}_a, \tilde{\nabla}_b](h_{cd} v^d) &= -R^e{}_{cab}{}^{(0)} (h_{ed} v^d) & (4.1)\\
\tilde{\nabla}_a \tilde{\nabla}_b h_{cd} v^d + \cancel{\tilde{\nabla}_b h_{cd} \tilde{\nabla}_a v^d + \tilde{\nabla}_a h_{cd} \tilde{\nabla}_b v^d} + h_{cd} \tilde{\nabla}_a \tilde{\nabla}_b v^d - (a \leftrightarrow b) &= -R^e{}_{cab}{}^{(0)} (h_{ed} v^d) & (4.2)\\
[\tilde{\nabla}_a, \tilde{\nabla}_b] h_{cd} v^d + h_{ce} [\tilde{\nabla}_a, \tilde{\nabla}_b] v^e &= -R^e{}_{cab}{}^{(0)} (h_{ed} v^d) & (4.3)\\
[\tilde{\nabla}_a, \tilde{\nabla}_b] h_{cd} v^d + h_{ce} R^e{}_{dab}{}^{(0)} v^d &= -R^e{}_{cab}{}^{(0)} h_{ed} v^d & (4.4)\\
\boxed{[\tilde{\nabla}_a, \tilde{\nabla}_b] h_{cd} = -R^e{}_{cab}{}^{(0)} h_{ed} - R^e{}_{dab}{}^{(0)} h_{ce}} & & (4.5)
\end{aligned}
$$

4.2 Lorenz Gauge of the Trace-reverse of Metric Perturbation

In Lorenz gauge, $0 = \tilde{\nabla}^a \bar{h}_{ab} = \tilde{\nabla}^a h_{ab} - \frac{1}{2} g_{ab} \tilde{\nabla}^a h$ in $3+1$ dimensions with a Ricci-flat background

$$G_{ab} = -\frac{1}{2}\lambda\left(\tilde{\nabla}_a\tilde{\nabla}_b h - \tilde{\nabla}^e\tilde{\nabla}_{(a}h_{b)e} + \tilde{\nabla}_d\tilde{\nabla}^d h_{ab} - g_{ab}\tilde{\nabla}_d\tilde{\nabla}^d h + g_{ab}\tilde{\nabla}^c\tilde{\nabla}^d h_{cd}\right) + \mathcal{O}(\lambda^2) \tag{4.6}$$

$$= -\frac{1}{2}\lambda\left(\tilde{\nabla}_a\tilde{\nabla}_b h - \tilde{\nabla}^e\tilde{\nabla}_{(a}h_{b)e} + \tilde{\nabla}_d\tilde{\nabla}^d h_{ab} - g_{ab}\tilde{\nabla}_d\tilde{\nabla}^d h + \frac{1}{2}g_{ab}\tilde{\nabla}^c(g_{cd}\tilde{\nabla}^d h)\right) + \mathcal{O}(\lambda^2) \tag{4.7}$$

$$= -\frac{1}{2}\lambda\left(\tilde{\nabla}_a\tilde{\nabla}_b h - \tilde{\nabla}^e\tilde{\nabla}_{(a}h_{b)e} + \tilde{\nabla}_d\tilde{\nabla}^d h_{ab} - \frac{1}{2}g_{ab}\tilde{\nabla}_d\tilde{\nabla}^d h\right) + \mathcal{O}(\lambda^2) \tag{4.8}$$

$$= -\frac{1}{2}\lambda\left(\tilde{\nabla}_a\tilde{\nabla}_b h - \tilde{\nabla}^e\tilde{\nabla}_a h_{be} - \tilde{\nabla}^e\tilde{\nabla}_b h_{ae} + \tilde{\nabla}_d\tilde{\nabla}^d \bar{h}_{ab}\right) + \mathcal{O}(\lambda^2) \tag{4.9}$$

$$= -\frac{1}{2}\lambda\left(\tilde{\nabla}_a\tilde{\nabla}_b h - \tilde{\nabla}^e\tilde{\nabla}_a\left(\bar{h}_{be} + \frac{1}{2}g_{be}h\right) - \tilde{\nabla}^e\tilde{\nabla}_b\left(\bar{h}_{ae} + \frac{1}{2}g_{ae}h\right) + \tilde{\nabla}_d\tilde{\nabla}^d \bar{h}_{ab}\right) + \mathcal{O}(\lambda^2) \tag{4.10}$$

$$= -\frac{1}{2}\lambda\left(\tilde{\nabla}_a\tilde{\nabla}_b h - \tilde{\nabla}^e\tilde{\nabla}_a\bar{h}_{be} - \frac{1}{2}\tilde{\nabla}_b\tilde{\nabla}_a h - \tilde{\nabla}^e\tilde{\nabla}_b\bar{h}_{ae} - \frac{1}{2}\tilde{\nabla}_a\tilde{\nabla}_b h + \tilde{\nabla}_d\tilde{\nabla}^d \bar{h}_{ab}\right) + \mathcal{O}(\lambda^2) \tag{4.11}$$

$$= -\frac{1}{2}\lambda\left(-\tilde{\nabla}^e\tilde{\nabla}_{(a}\bar{h}_{b)e} + \tilde{\nabla}_d\tilde{\nabla}^d \bar{h}_{ab}\right) + \mathcal{O}(\lambda^2) \tag{4.12}$$

$$= -\frac{1}{2}\lambda\left(-g^{ec}\left([\tilde{\nabla}_c, \tilde{\nabla}_a]\bar{h}_{be} + (a \leftrightarrow b)\right) + \tilde{\Box}\bar{h}_{ab}\right) + \mathcal{O}(\lambda^2) \tag{4.13}$$

$$= -\frac{1}{2}\lambda\left(-g^{ec}\left(-R^{d}{}_{bca}{}^{(0)}\bar{h}_{de} - R^{d}{}_{eca}{}^{(0)}\bar{h}_{bd} + (a \leftrightarrow b)\right) + \tilde{\Box}\bar{h}_{ab}\right) + \mathcal{O}(\lambda^2) \tag{4.14}$$

$$= -\frac{1}{2}\lambda\left(+\left(R^{d\ e}{}_{b\ a}{}^{(0)}\bar{h}_{de} + \cancel{R^{d\ e}{}_{e\ a}{}^{(0)}\bar{h}_{bd}} + (a \leftrightarrow b)\right) + \tilde{\Box}\bar{h}_{ab}\right) + \mathcal{O}(\lambda^2) \tag{4.15}$$

$$\boxed{G_{ab} = -\frac{1}{2}\lambda\left(2R^{c\ d}{}_{a\ b}{}^{(0)}\bar{h}_{cd} + \tilde{\Box}\bar{h}_{ab}\right) + \mathcal{O}(\lambda^2)} \tag{4.16}$$

Note that for WLP gauge that we choose later, $h = 0$, so $\bar{h}_{ab} = h_{ab}$.

4.3 Infinitesimal Gauge Transformation

We see that infinitesimal diffeomorphism $x^a \mapsto x'^{a'} = x^{a'} + \kappa^{a'}$, is equivalent to an infinitesimal gauge transformation of the metric at linear order:

$$g^{ab}(x) \mapsto g^{a'b'}(x') \tag{4.17}$$

$$= \frac{\partial x'^{a'}}{\partial x^a}\frac{\partial x'^{b'}}{\partial x^b} g^{ab}(x) \tag{4.18}$$

$$= (\delta^{a'}_a + \partial_a \kappa^{a'})(\delta^{b'}_b + \partial_b \kappa^{b'}) g^{ab}(x) \tag{4.19}$$

$$= \left(\delta^{a'}_a \delta^{b'}_b + \delta^{a'}_a \partial_b \kappa^{b'} + \partial_a \kappa^{a'} \delta^{b'}_b + \mathcal{O}(\kappa^2)\right) g^{ab}(x) \tag{4.20}$$

$$= g^{a'b'}(x) + \partial^{a'}\kappa^{b'} + \partial^{b'}\kappa^{a'} + \mathcal{O}(\kappa^2) \tag{4.21}$$

Therefore for first order perturbations, $h_{ab} \mapsto h_{ab} + \nabla^{(0)}_a \kappa_b + \nabla^{(0)}_b \kappa_a$ is a gauge transformation for arbitrary infinitesimal covector field κ_a.

We see that for the 10 components of h_{ab}, we have 4 gauge degrees of freedom. The remaining 6 are 2 propagating degrees of freedom and 4 static components.

5 Decoupling Limit of Scalar field

The action for an interacting scalar field (e.g. dynamical Chern-Simons) is

$$I = \int d^4x\sqrt{-g}\left[\frac{m_p^2}{2}R - \frac{1}{2}\partial_a\theta\partial^a\theta + \epsilon\mathcal{L}_{\text{int}}\right] \tag{5.1}$$

Imposing the principle of stationary action,

$$
\begin{aligned}
0 &= \delta I && (5.2)\\
&= \int\left\{\delta\sqrt{-g}\left[\frac{m_p^2}{2}R - \frac{1}{2}\partial_a\theta\partial^a\theta + \epsilon\mathcal{L}_{\text{int}}\right] + \sqrt{-g}\delta\left[\frac{m_p^2}{2}R - \frac{1}{2}\partial_a\theta\partial^a\theta + \epsilon\mathcal{L}_{\text{int}}\right]\right\}d^4x && (5.3)\\
&= \int d^4x\sqrt{-g}\left\{-\frac{1}{2}g_{ab}\delta g^{ab}\left[\frac{m_p^2}{2}R - \frac{1}{2}\partial_c\theta\partial^c\theta + \epsilon\mathcal{L}_{\text{int}}\right] + \frac{m_p^2}{2}\delta R + \delta\left[-\frac{1}{2}\partial_c\theta\partial^c\theta + \epsilon\mathcal{L}_{\text{int}}\right]\right\} && (5.4)\\
&= \int d^4x\sqrt{-g}\left\{-\frac{1}{2}g_{ab}\delta g^{ab}\left[\frac{m_p^2}{2}R - \frac{1}{2}\partial_c\theta\partial^c\theta + \epsilon\mathcal{L}_{\text{int}}\right] + \frac{m_p^2}{2}R_{ab}\delta g^{ab} - \frac{1}{2}\delta(\partial_c\theta\partial^c\theta) + \delta\left[\epsilon\mathcal{L}_{\text{int}}\right]\right\} && (5.5)\\
&= \int d^4x\sqrt{-g}\delta g^{ab}\left\{\frac{m_p^2}{2}\left(R_{ab} - \frac{1}{2}g_{ab}R\right) - \frac{1}{2}g_{ab}\left[-\frac{1}{2}\partial_c\theta\partial^c\theta + \epsilon\mathcal{L}_{\text{int}}\right]\right. && (5.6)\\
&\quad \left. -\frac{1}{2}\frac{\delta}{\delta g^{ab}}(g^{cd}\partial_c\theta\partial_d\theta) + \frac{\delta}{\delta g^{ab}}(\epsilon\mathcal{L}_{\text{int}})\right\} + \int d^4x\sqrt{-g}\left\{-\frac{\frac{1}{2}\delta(\partial_c\theta\partial^c\theta)}{\delta\theta}\delta\theta + \epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\delta\theta\right\} && (5.7)\\
&= \int d^4x\sqrt{-g}\delta g^{ab}\left\{\frac{m_p^2}{2}G_{ab} - \frac{1}{2}g_{ab}\left[-\frac{1}{2}\partial_c\theta\partial^c\theta + \epsilon\mathcal{L}_{\text{int}}\right] - \frac{1}{2}\delta_a^c\delta_b^d\partial_c\theta\partial_d\theta + \epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right\} && (5.8)\\
&\quad + \int d^4x\sqrt{-g}\left\{-\frac{\partial_c\theta\partial^c\delta\theta}{\delta\theta}\delta\theta + \epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\delta\theta\right\} && (5.9)\\
&= \frac{1}{2}\int d^4x\sqrt{-g}\delta g^{ab}\left\{m_p^2G_{ab} - \left[\partial_a\theta\partial_b\theta - \frac{1}{2}g_{ab}\partial_c\theta\partial^c\theta\right] + \epsilon\mathcal{L}_{\text{int}}g_{ab} + 2\epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right\} && (5.10)\\
&\quad + \int d^4x\delta\theta\left\{+\partial^c(\sqrt{-g}\nabla_c\theta) + \epsilon\sqrt{-g}\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right\} && (5.11)\\
0 &= \frac{1}{2}\int d^4x\sqrt{-g}\delta g^{ab}\left\{m_p^2G_{ab} - \left[\partial_a\theta\partial_b\theta - \frac{1}{2}g_{ab}\partial_c\theta\partial^c\theta\right] + \epsilon\mathcal{L}_{\text{int}}g_{ab} + 2\epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right\} && (5.12)\\
&\quad + \int d^4x\sqrt{-g}\delta\theta\left\{+\nabla^c\nabla_c\theta + \epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right\} && (5.13)
\end{aligned}
$$

Therefore our equations of motion are:

$$m_p{}^2G_{ab} + \underbrace{\epsilon\mathcal{L}_{\text{int}}g_{ab} + 2\epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}}_{\epsilon C_{ab}} = \underbrace{\partial_a\theta\partial_b\theta - \frac{1}{2}g_{ab}\partial_c\theta\partial^c\theta}_{T_{ab}{}^{(\theta)}} \tag{5.14}$$

$$\Box\theta = \underbrace{-\epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}}_{S} \tag{5.15}$$

We have the perturbative expansion from a Ricci-flat, scalarless background:

$$\theta = 0 + \epsilon\theta^{(1)} + \frac{1}{2}\epsilon^2\theta^{(2)} + \mathcal{O}(\epsilon^3) \tag{5.16}$$

$$g_{ab} = {g_{ab}}^{(0)} + \epsilon {h_{ab}}^{(1)} + \frac{1}{2}\epsilon^2 {h_{ab}}^{(2)} + \mathcal{O}(\epsilon^3) \tag{5.17}$$

$${T_{ab}}^{(\theta)} = \mathcal{O}(\epsilon^2) \tag{5.18}$$

$$R_{abcd} = \mathcal{O}(1) \tag{5.19}$$

$$\mathcal{L}_{\text{int}} = \mathcal{O}(\epsilon) \tag{5.20}$$

$$S = \mathcal{O}(\epsilon) \tag{5.21}$$

$$\epsilon C_{ab} = \mathcal{O}(\epsilon^2) \tag{5.22}$$

$$G_{ab} = -\frac{1}{2}\epsilon\left(2{R^{c\ \ d}_{\ a\ b}}^{(0)}{\bar{h}_{cd}}^{(1)} + \Box^{(0)}{\bar{h}_{ab}}^{(1)}\right) + \mathcal{O}(\epsilon^2) \tag{5.23}$$

So in the decoupling limit of $\epsilon \to 0$,

5.1 Zeroth Order

Just the Kerr solution with no scalar.

5.2 First Order

$$\Box^{(0)}\left(\epsilon\theta^{(1)}\right) = -\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right)^{(0)} \tag{5.24}$$

$$\Box^{(0)}\theta^{(1)} = -\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right)^{(0)} \tag{5.25}$$

and

$$m_p^2 G_{ab}^{(1)} + \epsilon\mathcal{L}_{\text{int}}^{(0)} g_{ab}^{(0)} + 2\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right)^{(0)} = 0 \tag{5.26}$$

$$m_p^2 G_{ab}^{(1)} = 0 \tag{5.27}$$

$$\left(2{R^{c\ \ d}_{\ a\ b}}^{(0)} + \delta_a^c\delta_b^d\Box^{(0)}\right){\bar{h}_{cd}}^{(1)} = 0 \tag{5.28}$$

where a solution is ${\bar{h}_{cd}}^{(1)} = 0$.

5.3 Second Order

Now at $\mathcal{O}(\epsilon^2)$ order, assuming ${\bar{h}_{cd}}^{(1)} = 0$,

$$m_p^2 G_{ab}^{(2)} + \epsilon\mathcal{L}_{\text{int}}^{(1)} g_{ab}^{(0)} + 2\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right)^{(1)} = \partial_a\left(\epsilon\theta^{(1)}\right)\partial_b\left(\epsilon\theta^{(1)}\right) - \frac{1}{2}g_{ab}^{(0)}\partial_c\left(\epsilon\theta^{(1)}\right)\partial^c\left(\epsilon\theta^{(1)}\right) \tag{5.29}$$

which reduces to

$$G_{ab}^{(2)} = \underbrace{m_p^{-2}\left[\underbrace{-\epsilon\mathcal{L}_{\text{int}}^{(1)}g_{ab}^{(0)} - 2\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right)^{(1)}}_{-\epsilon C_{ab}^{(1)}} + \underbrace{\epsilon^2\partial_a\theta^{(1)}\partial_b\theta^{(1)} - \frac{1}{2}\epsilon^2 g_{ab}^{(0)}\partial_c\theta^{(1)}\partial^c\theta^{(1)}}_{T_{ab}^{(2)}}\right]}_{S_{ab}^{(2)}} \tag{5.30}$$

$$\implies -\frac{1}{2(2!)}\left(2R^{c\ \ d\ (0)}_{\ a\ \ b} + \delta_a^c\delta_b^d\Box^{(0)}\right)\bar{h}_{cd}^{(2)} = S_{ab}^{(2)} \tag{5.31}$$

5.4 Third Order

We need to find θ to second order in ϵ:

$$\Box^{(0)}\left(\frac{1}{2}\epsilon^2\theta^{(2)}\right) = -\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right)^{(2)} \tag{5.32}$$

$$\Box^{(0)}\theta^{(2)} = -\frac{2}{\epsilon}\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right)^{(2)} \tag{5.33}$$

Then we have to $\mathcal{O}(\epsilon^3)$ order, assuming $\bar{h}_{cd}^{(1)} = 0$,

$$G_{ab}^{(3)} = \underbrace{m_p^{-2}\left[\underbrace{-\epsilon\mathcal{L}_{\text{int}}^{(2)}g_{ab}^{(0)} - 2\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right)^{(2)}}_{-\epsilon C_{ab}^{(2)}} + \underbrace{\frac{1}{2}\epsilon^2\left(\partial_a\theta^{(1)}\partial_b\theta^{(2)} + \partial_a\theta^{(2)}\partial_b\theta^{(1)} - g_{ab}^{(0)}\partial_c\theta^{(1)}\partial^c\theta^{(2)}\right)}_{T_{ab}^{(3)}}\right]}_{S_{ab}^{(3)}} \tag{5.34}$$

$$\implies -\frac{1}{2(3!)}\left(2R^{c\ \ d\ (0)}_{\ a\ \ b} + \delta_a^c\delta_b^d\Box^{(0)}\right)\bar{h}_{cd}^{(3)} = S_{ab}^{(3)} \tag{5.35}$$

5.5 Observation

We see as expected, the part of each order of G_{ab} acting on the solely the highest derivative of the metric is always an operator of the form $2R^{c\ \ d\ (0)}_{\ a\ \ b} + \delta_a^c\delta_b^d\Box^{(0)}$. This comes from the product of the perturbation expansion always has the same form for tems that have a single combinatorial contribution.

[consider expanding]

6 Inner Product Space of Perturbations

A natural first attempt at an inner product of p_{ab}, q_{cd} in the space of first order stationary, axisymmmetric perturbations of a background metric $g_{ab}^{(0)}$ is

$$\langle p, q\rangle \equiv \int p^{ab}q_{ab}\sqrt{g_{(0)}}d^4x \tag{6.1}$$

$$\langle p, q\rangle = \int dt\, d\phi \int p_{ab}\, g_{(0)}^{ac}\, g_{(0)}^{bd}\, q_{cd}\sqrt{g_{(0)}}d^2x \tag{6.2}$$

where raising and lowering is done by the background metric. Note that in equation (6.2) is only true for stationary, axisymmetric, metrics. The t and ϕ integrals are always the same for all p_{ab} and q_{cd}, so we can factor it out of all inner products.

6.1 Trace-reverse and the Inner Product

As a reminder, $\overline{\overline{p}}_{ab} = p_{ab}$, because $\left(p_{ab} - \frac{2}{d} g^{(0)}_{ab} p\right) - \frac{2}{d} g^{(0)}_{ab} \left(p - \frac{2}{d} g^{(0)}_{ab} g^{ab}_{(0)} p\right) = p_{ab}$ and that

$$\bar{p}^{ab}\bar{q}_{ab} = \left(p^{ab} - \frac{2}{d} g^{ab}_{(0)} p\right)\left(q_{ab} - \frac{2}{d} g^{(0)}_{ab} q\right) \tag{6.3}$$

$$= p^{ab}q_{ab} - \frac{2}{d}pq - \frac{2}{d}pq + \frac{4}{d}\frac{g^{ab}_{(0)} g^{(0)}_{ab}}{d} pq \tag{6.4}$$

$$= p^{ab}q_{ab} \tag{6.5}$$

$$\implies \langle p, q\rangle = \langle \bar{p}, \bar{q}\rangle \tag{6.6}$$

6.2 Self-Adjointness of the Linearized Einstein Operator

Reading off the form of the linearized Einstein operator $G^{(1)}$ in Lorenz gauge from eq. (5.28),

$$\langle p, G^{(1)}[q]\rangle = \int d^4x \sqrt{g_{(0)}}\, p^{ab} G^{(1)}[q]_{ab} \tag{6.7}$$

$$= \int d^4x \sqrt{g_{(0)}}\, p^{ab}\left(2R^{c\ \ d\ (0)}_{\ a\ \ b} + \delta^c_a \delta^d_b \Box^{(0)}\right)\bar{q}_{cd} \tag{6.8}$$

$$= \int d^4x \sqrt{g_{(0)}}\left(2R^{a\ \ b\ (0)}_{\ c\ \ d} p_{ab}\bar{q}^{cd} + p^{cd}\Box^{(0)}\bar{q}_{cd}\right) \tag{6.9}$$

$$= \int d^4x \sqrt{g_{(0)}}\left(\overline{2R^{a\ \ b\ (0)}_{\ c\ \ d} p_{ab}} q^{cd} + \bar{p}^{cd}\Box^{(0)} q_{cd}\right) \tag{6.10}$$

$$= \int d^4x \sqrt{g_{(0)}}\left(2R^{a\ \ b\ (0)}_{\ c\ \ d} \bar{p}_{ab} q^{cd} + \bar{p}^{cd}\Box^{(0)} q_{cd}\right) \tag{6.11}$$

where the last step is because we have a Ricci-flat background, so $R^{a\ \ b\ (0)}_{\ c\ \ d} g^{(0)}_{ab} = 0 = R^{a\ \ b\ (0)}_{\ c\ \ d} g^{cd}_{(0)}$. And in general, we see that the trace-reverse operator commutes with $G^(1)$, i.e. for all q, $\overline{G^{(1)}[\bar{q}]} = G^{(1)}[q]$.

Examining the second term of the integral, we integrate by parts twice and make use of the use the identity (A.18),

$$\int d^4x \sqrt{g_{(0)}}\, \bar{p}^{cd}\tilde{\nabla}_a\tilde{\nabla}^a q_{cd} = \int d^4x \sqrt{g_{(0)}}\, \tilde{\nabla}_a(\bar{p}^{cd}\tilde{\nabla}^a q_{cd}) - \int d^4x \sqrt{g_{(0)}}\, \tilde{\nabla}_a \bar{p}^{cd}\tilde{\nabla}^a q_{cd} \tag{6.12}$$

$$= \int d^4x\, \partial_a(\sqrt{g_{(0)}}\bar{p}^{cd}\tilde{\nabla}^a q_{cd}) - \int d^4x \sqrt{g_{(0)}}\, \tilde{\nabla}_a \bar{p}^{cd}\tilde{\nabla}^a q_{cd} \tag{6.13}$$

$$= -\int d^4x \sqrt{g_{(0)}}\, \tilde{\nabla}^a(\tilde{\nabla}_a \bar{p}^{cd} q_{cd}) + \int d^4x \sqrt{g_{(0)}}\, \tilde{\nabla}^a\tilde{\nabla}_a \bar{p}^{cd} q_{cd} \tag{6.14}$$

$$= -\int d^4x\, \partial_a(\sqrt{g_{(0)}}\tilde{\nabla}^a \bar{p}^{cd} q_{cd}) + \int d^4x \sqrt{g_{(0)}}\, \tilde{\nabla}^a\tilde{\nabla}_a \bar{p}^{cd} q_{cd} \tag{6.15}$$

Therefore, we have

$$\langle p, G^{(1)}[q]\rangle = \int d^4x \sqrt{g_{(0)}}\left(2R^{a\ \ b\ (0)}_{\ c\ \ d} + \delta^a_c \delta^b_d \Box^{(0)}\right)\bar{p}_{ab} q^{cd} \tag{6.16}$$

$$= \int d^4x \sqrt{g_{(0)}}\, G^{(1)}[p]^{cd} q_{cd} \tag{6.17}$$

$$= \langle G^{(1)}[p], q\rangle \tag{6.18}$$

The operator $G^{(1)}$ is self-adjoint with respect to this inner product.

7 Birkhoff's Theorem

Here is a nice (full) proof of Birkhoff's theorem. The main idea comes from Eric Poisson.[1]

7.1 Spherical Symmetry

Assuming a spherically symmetric $3+1$ dimensional spacetime, we can choose coordinates so that the metric has the general form:

$$ds^2 = A(t,r)dt^2 + B(t,r)dt\,dr + C(t,r)dr^2 + D(t,r)d\Omega^2 \tag{7.1}$$

We can transform our coordinates (t,r) so that r becomes $\sqrt{D}$. We choose the positive root because we want the angular coordinates to have positive Lorentzian signature (If we choose the negative convention our final metric change to reflect the convention change). Therefore we can always rewrite our spherically symmetric metric as

$$ds^2 = A(t,r)dt^2 + B(t,r)dt\,dr + C(t,r)dr^2 + r^2d\Omega^2 \tag{7.2}$$

where we have chosen the coordinate r specifically to give the spatial 2-sphere an r^2 areal dependence in the 4-fold.

Given any $A(t,r), B(t,r), C(t,r)$, we can transform the t coordinates so that our new coordinates, $t'(t,r)$ and r, gives

$$dt'^2 = \left(\frac{\partial t'}{\partial t}dt + \frac{\partial t'}{\partial r}dr\right)^2 \tag{7.3}$$

$$dt'^2 = \left(\frac{\partial t'}{\partial t}\right)^2 dt^2 + 2\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}dtdr + \left(\frac{\partial t'}{\partial r}\right)^2 dr^2 \tag{7.4}$$

$$D(t',r)\left(\frac{\partial t'}{\partial t}\right) = A(t,r) \tag{7.5}$$

$$D(t',r)\left(2\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}\right) = B(t,r) \tag{7.6}$$

$$E(t',r) - D(t',r)\left(\frac{\partial t'}{\partial r}\right)^2 = C(t,r) \tag{7.7}$$

Since we have three equations for three variables $t'(t,r), D\left(t'(t,r),r\right), E\left(t'(t,r),r\right)$, the equations are always soluble up given initial conditions. The choice of initial conditions is part of the gauge choice of our coordinate system. Then the line element is

$$ds^2 = D(t,r)dt^2 + E(t,r)dr^2 + r^2d\Omega^2 \tag{7.8}$$

We see that we have two functional degrees of freedom assuming spherical symmetry. Once the vacuum Einstein Field Equations are imposed, we will see that only a real valued parameter will remain as a degree of freedom.

7.2 Vacuum Einstein Field Equations

In regions where D and E do not blow up or go to 0, we can renaming our metric degrees of freedom, in two steps:

$$ds^2 = -e^{2\psi(t,r)}f(t,r)dt^2 + \frac{1}{f(t,r)}dr^2 + r^2d\Omega^2 \tag{7.9}$$

$$ds^2 = -e^{2\psi(t,r)}\left(1 - \frac{2m(t,r)}{r}\right)dt^2 + \left(1 - \frac{2m(t,r)}{r}\right)^{-1}dr^2 + r^2d\Omega^2 \tag{7.10}$$

In complete vacuum ${T^\mu}_\nu = 0$, we have that for the Einstein tensor ${G^\mu}_\nu$ with the help of Mathematica,

$$0 = {G^t}_t = \frac{-2\partial_r m(t,r)}{r^2} \tag{7.11}$$

$$0 = {G^r}_t = \frac{2\partial_t m(t,r)}{r^2} \tag{7.12}$$

$$0 = {G^r}_r - {G^t}_t = \frac{2}{r}\left(1 - \frac{2m(t,r)}{r}\right)\partial_r\psi(t,r) \tag{7.13}$$

By equation (7.11), $m(t,r) = m(t)$ and by equation (7.12), $m(t,r) = m(r)$. Therefore $m(t,r)$ is a real constant.
Now by equation (7.13), we have $\psi(t,r) = \psi(t)$.
We can then rescale $t \mapsto e^{-\psi(t)}t$, so that $g_{tt} = -\left(1 - \frac{2m}{r}\right)$ and all other metric components stay the same.

Therefore the unique spherically symmetric solution to the vacuum Einstein Field equations with $\Lambda = 0$ is the Schwarzschild solution:

$$\boxed{ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2 d\Omega^2} \tag{7.14}$$

for some coordinates with the $- + ++$ Lorentzian signature.

Notice we see that any spherically symmetric solution must be asymptotically flat (as $r \to \infty$) and static (with respect to the time-like vector $\frac{\partial}{\partial t}$); we did not impose these conditions.

Therefore, there is no gravitational monopole radiation.

7.3 Komar Mass

It turns out the Komar mass integral of the Schwarzschild solution is m, so m really does correspond to a physical mass of the metric.

8 Weyl-Lewis-Papapetrou

In order to prove this we need a little machinery called Frobenius' Theorem.

8.1 Frobenius' Theorem

There are a few equivalent statements of Frobenius' Theorem; while the differential form version is nice, we use the vector field form for our current purposes. Frobenius' Theorem is useful not only for the proof of uniqueness of the WLP metric, but also will be used to show the integrability conditions for the solution to the Einstein Field Equations under a WLP metric.

Without introducing to many definitions, the theorem is roughly

Theorem 8.1 *In order to have a smooth sub-manifold of $\mathcal{M}$ that has tangent spaces coinciding with a tangent sub-bundle $W \subseteq E$ over $\mathcal{M}$, it is necessary and sufficient for W to be involute, i.e. $\forall X^a, Y^a \in W : [X,Y]^a \in W$.*

Therefore we have the following corollary:

Corollary 8.1.1 *If vector fields X^a and Y^a commute, with either vanishing at a point, and*

$$X^a R_a{}^{[b} X^c Y^{d]} = 0 = Y^a R_a{}^{[b} Y^c X^{d]}, \tag{8.1}$$

then the 2-fold orthogonal to X^a and Y^a are integrable.

The proofs are outlined in Wald[2], and may be reproduced here at a later time.

8.2 Proof of WLP

Given a time-like $\left(\frac{\partial}{\partial t}\right)^a$ and an "azimuthal" space-like $\left(\frac{\partial}{\partial \phi}\right)^a$ Killing vector fields for stationary axisymmetric $1+3$ dimensional spacetimes. Assuming these satisfy corollary 8.1.1, the span of the other vector fields generated by the other two coordinates (x_2 and x_3) are orthogonal to ∂_t^a and ∂_ϕ^a. (The first condition of corollary 8.1.1 is trivial, but for the second there is a possible argument based on t- and ϕ-reversal symmetry, but further investigation is needed.)

$$ds^2 = V(x_2,x_3)dt^2 + 2W(x_2,x_3)dtd\phi + X(x_2,x_3)d\phi^2 + g_{ij}(x_2,x_3)dx^i dx^j \tag{8.2}$$

for $i, j \in \{2, 3\}$. In block matrix form, the metric is

$$g_{ab} = \begin{pmatrix} -V & W & 0 & 0 \\ W & X & 0 & 0 \\ 0 & 0 & g_{22} & g_{23} \\ 0 & 0 & g_{23} & g_{33} \end{pmatrix} \tag{8.3}$$

Note that there are six distinct functions of x_2 and x_3.

We choose $x_2 = \rho = VX + W^2$, which is the negative of determinant of the upper 2×2 block. And choose $x_3 = z$ be such that $\nabla_a \rho \nabla^a z = 0$. Redefining variables, we must have

$$ds^2 = -V(dt - wd\phi)^2 + V^{-1}\rho^2 d\phi^2 + \Omega^2(d\rho^2 + \Lambda dz^2) \tag{8.4}$$

where $w = W/V$, $\Omega^2 = g_{22}$, and $\Lambda = g_{33}/\Omega^2$.

The four functional degrees of freedom are $V(\rho,z), w(\rho,z), \Omega(\rho,z), \Lambda(\rho,z)$.

We have made a gauge transformation to the unique Weyl-Lewis-Papapetrou coordinates for any stationary, axisymmetric spacetime, up to univariate scaling of z.

9 Schwarzschild in Weyl-Lewis-Papapetrou

9.1 Schwarzschild Background

We want to describe spacetimes in with a Schwarzschild background. Therefore we expect there to exist $V = V_0 + \delta V, w = w_0 + \delta w, \Omega = \Omega_0 + \delta\Omega, \Lambda = \Lambda_0 + \delta\Lambda$, where the variables with the naught-subscripts describe Schwarzschild background metric, and the δ variables are perturbations that keep the metric stationary and axisymmetric. Let's solve for the Schwarzschild solution only in terms of the background first, with no perturbations; we need to get the metric into the form:

$$ds^2 = -V_0(dt - w_0 d\phi)^2 + V_0^{-1}\rho^2 d\phi^2 + \Omega_0^2(d\rho^2 + \Lambda_0 dz^2) \tag{9.1}$$

Note that at the end of our calculation, we expect to choose coordinates so that $\Lambda_0 = 1$ because Schwarzschild is Ricci-flat.

9.2 Motivation of WLP Coordinates

By Birkhoff's Theorem, the Schwarzschild metric (7.14) is axisymmetric and stationary (in fact it is static):

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{9.2}$$

Therefore we should be able to write the metric in Weyl-Lewis-Papapetrou form.

We keep the time and azimuthal directions the same, as it is natural to pick $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ as our Killing vector fields. Therefore were are transforming the spatial coordinates r and θ only, from those that are spherically symmetric to those cylindrically symmetric.

We identify that $V_0 = 1 - \frac{2m}{r}$ and $w_0 = 0$, so our metric is in the form:

$$ds^2 = -V_0(dt - w_0 d\phi)^2 + V_0^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{9.3}$$

We see that the standard spherical to cylindrical ($r\sin\theta \mapsto \rho$, $r\cos\theta \mapsto z$) will not suffice because the only $d\phi^2$ term in the line element will be $r^2\sin^2\theta d\phi^2 \mapsto \rho^2 d\phi^2$, and in the WLP form, we need $V_0^{-1}\rho^2 d\phi^2$. Thus, we make our transformation $V_0^{1/2} r\sin\theta \mapsto \rho$, so that $r^2\sin^2\theta d\phi^2 \mapsto V_0^{-1}\rho^2 d\phi^2$.

Our transformation is so far defined by

$$\rho = V_0^{1/2} r\sin\theta = \sqrt{r^2 - 2mr}\sin\theta \tag{9.4}$$

$$\implies d\rho = \frac{r-m}{V_0^{1/2} r}\sin\theta dr + \underbrace{V_0^{1/2} r\cos\theta}_{\tilde{\rho}}\, d\theta \tag{9.5}$$

We see that $\tilde{\rho} = V_0^{1/2} r\cos\theta$ is the trigonometric conjugate of $\rho = V_0^{1/2} r\sin\theta$ (i.e. $\tilde{\rho}^2 + \rho^2 = V_0 r^2$). And with a clever definition of z, we have

$$z = (r - m)\cos\theta \tag{9.6}$$

$$\implies dz = \cos\theta dr - \underbrace{(r-m)\sin\theta}_{\tilde{z}}\, d\theta \tag{9.7}$$

where $\tilde{z} = (r - m)\sin\theta$ is the trignometric conjugate of $z = (r - m)\cos\theta$.

We see a good sign that $\frac{\tilde{z}}{V_0^{1/2} r}$ appears in (9.5) and $\frac{\tilde{\rho}}{V_0^{1/2} r}$ appears in (9.7).

So with this transformation:

$$\boxed{\begin{aligned} &t = t && (9.8)\\ &\rho = V_0^{1/2} r\sin\theta = \sqrt{r^2-2mr}\sin\theta && (9.9)\\ &z = (r-m)\cos\theta && (9.10)\\ &\phi = \phi && (9.11)\end{aligned}}$$

we have

$$\boxed{\begin{aligned} &dt = dt && (9.12)\\ &d\rho = V_0^{-1/2} r^{-1}\tilde{z}dr + \tilde{\rho}\mathrm{d}z && (9.13)\\ &dz = V_0^{-1/2} r^{-1}\tilde{\rho}dr - \tilde{z}d\theta && (9.14)\\ &d\phi = d\phi && (9.15)\end{aligned}}$$

Therefore, we have in terms of the auxiliary variables $\tilde{\rho} = V_0^{1/2} r\cos\theta$ and $\tilde{z} = (r-m)\sin\theta$,

$$\implies \tilde{z}d\rho + \tilde{\rho}dz = V_0^{-1/2} r^{-1}(\tilde{z}^2 + \tilde{\rho}^2)dr \tag{9.16}$$

$$\implies \boxed{dr = \frac{V_0^{1/2} r}{\tilde{z}^2 + \tilde{\rho}^2}(\tilde{z}d\rho + \tilde{\rho}dz)} \tag{9.17}$$

$$\implies \tilde{\rho}d\rho - \tilde{z}dz = (\tilde{z}^2 + \tilde{\rho}^2)d\theta \tag{9.18}$$

$$\implies \boxed{d\theta = \frac{1}{\tilde{z}^2 + \tilde{\rho}^2}(\tilde{\rho}d\rho - \tilde{z}dz)} \tag{9.19}$$

Substituting into the metric,

$$ds^2 = -V_0(dt - w_0 d\phi)^2 + V_0^{-1}\rho^2 d\phi^2 + \cancel{V_0^{-1}}\frac{\cancel{V_0} r^2}{(\tilde{z}^2+\tilde{\rho}^2)^2}(\tilde{z}d\rho + \tilde{\rho}dz)^2 + r^2\frac{(\tilde{\rho}d\rho - \tilde{z}dz)^2}{(\tilde{z}^2+\tilde{\rho}^2)^2} \tag{9.20}$$

$$ds^2 = -V_0(dt - w_0 d\phi)^2 + V_0^{-1}\rho^2 d\phi^2 + \frac{r^2}{(\tilde{z}^2+\tilde{\rho}^2)^{\cancel{2}}}\left(\cancel{(\tilde{z}^2+\tilde{\rho}^2)}d\rho^2 + \cancel{(\tilde{z}^2+\tilde{\rho}^2)}dz^2\right) \tag{9.21}$$

$$ds^2 = -V_0(dt - w_0 d\phi)^2 + V_0^{-1}\rho^2 d\phi^2 + \frac{r^2}{\tilde{z}^2+\tilde{\rho}^2}\left(d\rho^2 + dz^2\right) \tag{9.22}$$

We see that we've chosen z correctly so that $\Lambda_0 = 1$ and

$$\Omega_0^2 = \frac{r^2}{\tilde{z}^2+\tilde{\rho}^2} = \frac{r^2}{(r^2-2mr+m^2)\sin^2\theta + (r^2-2mr)\cos^2\theta} \tag{9.23}$$

$$= \frac{r^2}{(r^2-2mr)+m^2\sin^2\theta} \tag{9.24}$$

Therefore we have for the Schwarzschild background

$$ds^2 = -V_0(dt - w_0 d\phi)^2 + V_0^{-1}\rho^2 d\phi^2 + \Omega_0^2\left(d\rho^2 + \Lambda_0 dz^2\right) \tag{9.25}$$

So our Weyl-Lewis-Papapetrou functional degrees of freedom are, as functions (r,θ),

$$\boxed{\begin{aligned} &V = \left(1 - \frac{2m}{r}\right) + \delta V && (9.26)\\ &w = 0 + \delta w && (9.27)\\ &\Omega^2 = \frac{r^2}{(r^2-2mr)+m^2\sin^2\theta} + \delta\Omega^2 && (9.28)\\ &\Lambda = 1 + \delta\Lambda && (9.29)\end{aligned}}$$

9.3 Coordinate Singularities of Background Schwarzschild

Despite the curvature singularity at $r = 0$, we have coordinate singularities when $\Omega_0^2 \to \infty$, i.e.

$$0 = r^2 - 2mr + m^2 \sin^2\theta \tag{9.30}$$
$$0 = (r-m)^2 - m^2\cos^2\theta \tag{9.31}$$
$$0 = (\underbrace{r - m + m\cos\theta}_{R_+})(\underbrace{r - m - m\cos\theta}_{R_-}) \tag{9.32}$$

With the auxiliary variables $R_\pm$, we rewrite our WLP functions with the substitution $r = \frac{1}{2}(R_+ + R_- + 2m)$:

$$\boxed{\begin{aligned} V &= \frac{R_+ + R_- - 2m}{R_+ + R_- + 2m} + \delta V && (9.33)\\ w &= 0 + \delta w && (9.34)\\ \Omega^2 &= \frac{(R_+ + R_- + 2m)^2}{4R_+R_-} + \delta\Omega^2 && (9.35)\\ \Lambda &= 1 + \delta\Lambda && (9.36)\end{aligned}}$$

$$\rho^2 + z^2 = (r^2 - 2mr)\sin^2\theta + (r-m)^2\cos^2\theta \tag{9.37}$$
$$= (r-m)^2 + m^2\cos^2\theta - m^2 \tag{9.38}$$
$$= (r - m \pm m\cos\theta)^2 - m^2 \mp 2(r-m)m\cos\theta \tag{9.39}$$
$$= R_\pm^2 - m^2 \mp 2mz \tag{9.40}$$
$$\implies \rho^2 + (z\pm m)^2 = R_\pm^2 \tag{9.41}$$
$$\implies \boxed{R_\pm = \sqrt{\rho^2 + (z\pm m)^2}} \tag{9.42}$$

and thus our WLP functions are now functions of (ρ, z).

The coordinate singularities corresponding to $R_\pm = 0$ are now at $(\rho, z) = (0, \pm m)$ for all t and ϕ.

We also have a coordinate singularity when $\rho \to 0$, so all the coordinate singularities are at the line $\rho = 0$ in the spacetime, which includes the $(\rho, z) = (0, \pm m)$ singularity as well.

10 Mathematica for perturbations of Kerr and Schwarzschild

I was able to calculate the Einstein operator in WLP coordinates for both a Kerr and Schwarzschild backgrounds. The Kerr solution in WLP form I used are from Jones and Wang[3]. The solutions with the explicit coordinates are too long to reproduce here in the progress report, but the abberviated ones are below.

10.1 Schwarzschild Background

10.2 Kerr Background

11 Bianchi Identity

11.1 General Connections

Baez and Muniain[4] outline an elegant proof of the Bianchi identity, reproduced here in detail. We will use the the Bianchi identity to show the geometric origin of the divergencelessness of the Einstein tensor and all possible source terms.

Given a fiber bundle $\pi : E \to \mathcal{M}$ and a connection D on $\mathcal{M}$, for any E-valued form $\eta = s_I \otimes \omega^I$ on $\mathcal{M}$, in local coordinates,

$$
\begin{aligned}
\mathrm{d}_D^2 \eta &= \mathrm{d}_D \left(D_\nu s_I \otimes \mathrm{d}x^\nu \wedge \mathrm{d}x^I \right) && (11.1)\\
&= D_\mu D_\nu s_I \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (11.2)\\
&= \frac{1}{2} \left[D_\mu, D_\nu \right] s_I \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (11.3)\\
&= \frac{1}{2} F_{\mu\nu} s_I \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (11.4)\\
&= F \wedge \eta && (11.5)
\end{aligned}
$$

Note that the exterior covariant derivative doesn't form a de Rham cohomology where $\mathrm{d}^2 = 0$ because the covariant derivative is not commutative, unlike the partial derivative. The *failure* to commute is the geometric curvature.

Therefore,

$$
\begin{aligned}
\mathrm{d}_D^3 \eta &= \mathrm{d}_D (\mathrm{d}_D^2 \eta) && (11.6)\\
&= \mathrm{d}_D (F \wedge \eta) && (11.7)\\
&= \mathrm{d}_D F \wedge \eta + F \wedge \mathrm{d}_D \eta && (11.8)\\
\mathrm{d}_D^3 \eta &= \mathrm{d}_D^2 (\mathrm{d}_D \eta) && (11.9)\\
&= F \wedge \mathrm{d}_D \eta && (11.10)\\
\implies & \boxed{\mathrm{d}_D F = 0} && (11.11)
\end{aligned}
$$

In local coordinates,

$$
\begin{aligned}
0 = \mathrm{d}_D F \wedge \eta = \mathrm{d}_D & \left(\frac{1}{2} F_{\mu\nu} \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \right) \wedge \left(s_I \otimes \mathrm{d}x^I \right) && (11.12)\\
&= \frac{1}{2} (D_\lambda F_{\mu\nu}) \otimes \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \left(s_I \otimes \wedge \mathrm{d}x^I \right) && (11.13)\\
&= \frac{1}{2} (D_\lambda F_{\mu\nu}) s_I \otimes \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (11.14)\\
&= \frac{1}{2} \left(D_\lambda \left(F_{\mu\nu} s_I \right) - F_{\mu\nu} \left(D_\lambda s_I \right) \right) \otimes \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (11.15)\\
&= \frac{1}{2} [D_\lambda, F_{\mu\nu}] s_I \otimes \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (11.16)\\
&= \frac{1}{2} \cdot \frac{1}{3} \left([D_\lambda, F_{\mu\nu}] + [D_\mu, F_{\nu\lambda}] + [D_\nu, F_{\lambda\mu}] \right) s_I \otimes \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (11.17)\\
\implies 0 &= [D_\lambda, F_{\mu\nu}] + [D_\mu, F_{\nu\lambda}] + [D_\nu, F_{\lambda\mu}] && (11.18)\\
& \boxed{0 = [D_\lambda, [D_\mu, D_\nu]] + [D_\mu, [D_\nu, D_\lambda]] + [D_\nu, [D_\lambda, D_\mu]]} && (11.19)
\end{aligned}
$$

which is in the form of the Jacobi identity.

11.2 With Riemann curvature

For our Levi-Civita connection ∇ compatible with metric g, we have the curvature

$$
R(u, v) w = \left([\nabla_u, \nabla_v] - \nabla_{[u,v]} \right) w, \tag{11.20}
$$

which is just the curvature of the connection ∇.

$$
\begin{aligned}
0 &= [u,[v,w]] + [v,[w,u]] + [w,[u,v]] && (11.21)\\
&= \nabla_u[v,w] - \nabla_{[v,w]}u + (uvw \text{ cyc}) && (11.22)\\
&= \nabla_u(\nabla_v w - \nabla_w v) - \nabla_{[v,w]}u + (uvw \text{ cyc}) && (11.23)\\
&= [\nabla_u, \nabla_v]w - \nabla_{[u,v]}w + (uvw \text{ cyc}) && (11.24)\\
0 &= R(\nabla_u, \nabla_v)w + (uvw \text{ cyc}) && (11.25)\\
& && (11.26)
\end{aligned}
$$

Specifically, the Riemann curvature tensor is $R^a{}_{bcd}e_a \equiv R(\nabla_b, \nabla_c)e_d$. Choose $u = \partial_a, v = \partial_b, w = \partial_c$ to be coordinate basis vector fields.

$$
\begin{aligned}
&\implies 0 = R(\nabla_a, \nabla_b)\partial_c + R(\nabla_b, \nabla_c)\partial_a + R(\nabla_c, \nabla_a)\partial_b && (11.27)\\
&\implies 0 = R^d{}_{abc} + (abc \text{ cyc}) && (11.28)\\
&\implies \boxed{0 = R^d{}_{[abc]}} && (11.29)
\end{aligned}
$$

From eq. (11.19) applied to the Levi-Civita connection,

$$
\begin{aligned}
0 &= [\nabla_a,[\nabla_b,\nabla_c]] + [\nabla_b,[\nabla_c,\nabla_a]] + [\nabla_c,[\nabla_a,\nabla_b]] && (11.30)\\
&= [\nabla_a, R(\nabla_b,\nabla_c)]e_d + (abc \text{ cyc}) && (11.31)\\
&= \nabla_a R^e{}_{bcd} e_e - \cancel{R(\nabla_b,\nabla_c)g_{ad}} + (abc \text{ cyc}) && (11.32)\\
\implies 0 &= \nabla_a R_{ebcd} + (abc \text{ cyc}) && (11.33)\\
\implies 0 &= \nabla_a R_{ecdb} + \nabla_a R_{edbc} + (abc \text{ cyc}) && (11.34)
\end{aligned}
$$

where we use eq. (11.28) in the last step.

Contracting with the metric twice,

$$
\begin{aligned}
0 &= g^{ec}\left(\cancel{\nabla_a R_{ecdb}} + \nabla_a R_{edbc} + (abc \text{ cyc})\right) && (11.35)\\
0 &= -\nabla_a R_{db} + \nabla_b R_{da} + \nabla^e R_{edab} && (11.36)\\
0 &= g^{bd}\left(-\nabla_a R_{db} + \nabla_b R_{da} + \nabla^e R_{edab}\right) && (11.37)\\
0 &= -\nabla_a R + \nabla^d R_{da} + \nabla^e R_{ea} && (11.38)\\
\implies 0 &= \nabla^d \underbrace{(2R_{da} - g_{da}R)}_{2G_{da}} && (11.39)\\
\implies & \boxed{0 = \nabla^d G_{da}} && (11.40)
\end{aligned}
$$

12 The action of Einstein operator in WLP gauge: Ricci-flat

[This part was quite difficult, even for with the Ricci-flat simplification. The manipulations here are not referenced anywhere and it took a lot of sweat and trial and error to get the following result.]

12.1 Constraint equations

From $G_{ab} = 0$, we have ostensibly 6 non-zero equations of motion, which correspond to G_{00}, G_{03}, G_{33} and G_{11}, G_{12}, G_{22}.

From the first three, we can construct the combinations

$$e^{2\gamma}\left(\left(V^{-2}-\rho^{-2}w^2\right)G_{00}+e^{2\gamma}\rho^{-2}w^2\right)G_{33}=\vec{\nabla}\left(V^{-1}\vec{\nabla}V+\rho^{-2}V^2w\vec{\nabla}w\right) \tag{12.1}$$

$$e^{2\gamma}\rho^{-2}\left(wG_{00}+G_{03}\right)=\vec{\nabla}\left(\rho^{-2}V^2\vec{\nabla}w\right) \tag{12.2}$$

where $\vec{\nabla}$ is the gradient under the flat metric $ds^2=\rho^2d\phi^2+d\rho^2+dz^2$, *not* $ds^2=g_{ab}dx^adx^b$

We have $G_{00}=G_{03}=G_{33}=0$ if and only if

$$0=\vec{\nabla}\cdot\left(V^{-1}\vec{\nabla}V+\rho^{-2}V^2w\vec{\nabla}w\right) \tag{12.3}$$

$$0=\vec{\nabla}\cdot\left(\rho^{-2}V^2\vec{\nabla}w\right) \tag{12.4}$$

and the Bianchi identity $\nabla^aG_{ab}=0$ is satisfied.

Furthermore, we have $G_{11}=-G_{22}$ automatically, so we are left with

$$0=-G_{11}=G_{22}=\frac{1}{4V^2}\left((\partial_\rho V)^2-(\partial_z V)^2\right)-\frac{V^2}{4\rho^2}\left((\partial_\rho w)^2-(\partial_z w)^2\right)-\frac{\partial_\rho\gamma}{\rho^2} \tag{12.5}$$

$$0=G_{12}=\frac{\partial_z\gamma}{\rho^2}-\frac{1}{2V^2}(\partial_\rho V)(\partial_z V)+\frac{V^2}{2\rho^2}(\partial_\rho w)(\partial_z w) \tag{12.6}$$

which are compatible because given eqs. (12.3) and (12.4), $\partial_\rho\partial_z\gamma=\partial_z\partial_\rho\gamma$ is true.

We have shown that there are 4 equations (2 of which are compatible) consistent with 3 metric variables in the Ricci-flat case.

13 Non-Ricci-flat Perturbations of Ricci-flat Background

Since we know $G_{ab}=T^{(0)}_{ab}+\epsilon T^{(1)}_{ab}+\mathcal{O}(\epsilon^2)$ and $T^{(0)}_{ab}=0$, for sake of brevity, we use the notation $T_{ab}\equiv T^{(1)}_{ab}$, so that for the order ϵ^1 term, $G^{(1)}_{ab}=T_{ab}$.

14 Linearized Einstein Field Equations of WLP perturbations

15 z Gauge Fixing

15.1 $z\mapsto f(z)$

We have a remaining gauge freedom in WLP, $z\mapsto f(z)$ keeps the metric in WLP form. We need to fix the gauge completely to perform explicit numerical calculations. The map $z\mapsto f(z)$ changes the WLP metric by

$$ds^2=-V(dt-wd\phi)^2+V^{-1}\left(\rho^2d\phi^2+e^{2\gamma}(d\rho^2+e^{2\lambda}dz^2)\right) \tag{15.1}$$

$$\mapsto -V(dt-wd\phi)^2+V^{-1}\left(\rho^2d\phi^2+e^{2\gamma}(d\rho^2+e^{2\lambda}\left(\partial_zf\right)^2dz^2)\right) \tag{15.2}$$

$$=-V(dt-wd\phi)^2+V^{-1}\left(\rho^2d\phi^2+e^{2\gamma}(d\rho^2+e^{2(\lambda+\log\partial_zf)}dz^2)\right) \tag{15.3}$$

So the gauge freedom is

$$\lambda\mapsto\lambda+\log\partial_zf \tag{15.4}$$

$$\implies \lambda_0+\epsilon\delta\lambda\mapsto\lambda_0+\epsilon\delta\lambda+\log\partial_zf \tag{15.5}$$

We first fix our gauge so that $\lambda_0 = 0$, so the remaining gauge freedom is, for any function $G(z)$ that is $\mathcal{O}(\epsilon)$,

$$\delta\lambda \mapsto \delta\lambda + \log \partial_z f \tag{15.6}$$

$$\partial_z \delta\lambda \mapsto \partial_z \delta\lambda + \underbrace{\frac{\partial_z^2 f}{\partial_z f}}_{G(z)} \tag{15.7}$$

which means once we fix our gauge with $G(z)$ completely we have the condition that

$$\partial_z \delta\lambda + G(z) = H(\rho, z) \tag{15.8}$$

for an *a priori* unknown function $H(\rho, z)$

From the six original linearized EFEs, and imposing the background Wald equations we have

$$\partial_\rho \delta\lambda = \rho(T_{11} - T_{22}) \tag{15.9}$$

$$\implies \partial_z \partial_\rho \delta\lambda = \rho \partial_z (T_{11} - T_{22}) \tag{15.10}$$

Taking the ρ partial derivative of eq. (15.8) yields,

$$\partial_\rho \partial_z \delta\lambda = \partial_\rho H \tag{15.11}$$

$$\implies \partial_\rho H = \rho \partial_z (T_{11} - T_{22}) \tag{15.12}$$

Assuming $H(\rho = R, z) = 0$, for some R (which could be ∞, we have

$$\implies H(\rho, z) = \int_R^\rho \rho' \partial_z \left(T_{11}(\rho', z) - T_{22}(\rho', z)\right) d\rho' + C(z) \tag{15.13}$$

for some arbitrary constant $C(z)$.

But this $C(z)$ degree of ambiguity for $H(\rho, z)$ is exactly the gauge degree of freedom $G(z)$ in eq. (15.8)! Therefore, we have

$$\partial_z \delta\lambda = \int_R^\rho \rho' \partial_z \left(T_{11}(\rho', z) - T_{22}(\rho', z)\right) d\rho' + \tilde{C}(z) \tag{15.14}$$

where $\tilde{C}(z) = C(z) - G(z)$.

For our numerical purposes, we can just set $\tilde{C}(z) = 0$ to completely fix our z gauge degree of freedom.

15.2 Flat Laplacian of $\delta\lambda$

Therefore we have explicitly, $\partial_\rho \delta\lambda$ and $\partial_z \delta\lambda$, so we can construct the flat laplacian of $\delta\lambda$ under the metric $ds^2 = \rho d\phi^2 + d\rho^2 + dz^2$,

$$\nabla^2 \delta\lambda = \left(\partial_\rho^2 + \frac{\partial_\rho}{\rho} + \partial_z^2\right) \delta\lambda = \int_\infty^\rho \rho' \partial_z^2 \left(T_{11}(\rho', z) + T_{22}(\rho', z)\right)\, d\rho' + \rho \partial_\rho \left(T_{11} + T_{22}\right) + 2(T_{11} + T_{22}) \tag{15.15}$$

along with the flat laplacians of $\delta V, \delta w$, and $\delta\gamma$ we found earlier.

16 Dynamical Chern-Simons over Kerr Background

Solve for eom

Solve for C_{ab}, T_{ab}

17 Numerics

We use non-minimally coupled scalar to the Pontryagin-Chern density, ${}^*RR \equiv -\frac{1}{2}\epsilon^{abcd}R_{abef}R_{cd}{}^{ef}$, over a Kerr background.

From eq. (5.25), we have the equation (with the conventional coupling factor of $\frac{1}{8}$ from [5])

$$\Box^{(0)}\theta^{(1)} = -\frac{1}{16}\epsilon^{abcd}R^{(0)}_{abef}R^{(0)ef}_{cd} \tag{17.1}$$

18 Maximum Principle Proof

19 Transformation to Rational-Polynomial Boyer-Lindquist Coordinates

For a $\lambda_0 = 0$, a Ricci-flat background, the background scalar laplacian is

$$\nabla^2_{\mathrm{WLP}} f = V_0 e^{-2\gamma_0}\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2\right) f(\rho, z) = \frac{1}{\Sigma}\left(\partial_r \Delta \partial_r + \partial_\mu (1-\mu^2)\partial_\mu\right) f(r,\mu) = \nabla^2_{\mathrm{BL}} \tag{19.1}$$

Challenges

- Non-Ricci Flat case
- Express constraints in terms of perturbations and background
- Check if ADM equations are automatically satisfied or need to be constrained during the numerical evolution.
- Invert Linearized equation
- Relaxation code and numerics
- boundary conditions, and compactifying coordinates to bring in infinity.

Appendices

A Miscellaneous Identities Used in Proofs

A.1 Metric

$$\begin{aligned}
\frac{\partial g^{\rho\sigma}}{\partial g_{\mu\nu}} &= \frac{\partial(g^{\rho\rho'}g^{\sigma\sigma'}g_{\rho'\sigma'})}{\partial g_{\mu\nu}} = \frac{\partial g^{\rho\rho'}}{\partial g_{\mu\nu}}g^{\sigma\sigma'}g_{\rho'\sigma'} + g^{\rho\rho'}\frac{\partial g^{\sigma\sigma'}}{\partial g_{\mu\nu}}g_{\rho'\sigma'} + g^{\rho\rho'}g^{\sigma\sigma'}\frac{\partial g_{\rho'\sigma'}}{\partial g_{\mu\nu}} \\
&= \frac{\partial g^{\rho\rho'}}{\partial g_{\mu\nu}}\delta^{\sigma}_{\rho'} + \frac{\partial g^{\sigma\sigma'}}{\partial g_{\mu\nu}}\delta^{\rho}_{\sigma'} + g^{\rho\rho'}g^{\sigma\sigma'}\delta^{\mu}_{\rho'}\delta^{\nu}_{\sigma'} \\
&= \frac{\partial g^{\rho\sigma}}{\partial g_{\mu\nu}} + \frac{\partial g^{\sigma\rho}}{\partial g_{\mu\nu}} + g^{\rho\mu}g^{\sigma\nu} \\
\implies & \boxed{\frac{\partial g^{\rho\sigma}}{\partial g_{\mu\nu}} = -g^{\rho\mu}g^{\sigma\nu}}
\end{aligned} \tag{A.1}$$

A.2 Jacobi Formula

For a generic derivative operator ∂, one can show the following two facts:

$$\log\det A = \operatorname{tr}\log A \tag{A.2}$$

$$\partial \operatorname{tr} F(A) = \operatorname{tr}\left(\frac{d}{dA}F(A)\partial A\right) \tag{A.3}$$

Then one can prove:

$$\begin{aligned}
\frac{1}{\det A}\partial \det A = \partial \log\det A &= \partial \operatorname{tr}\log A \\
&= \operatorname{tr}\left(\frac{d}{dA}\log A\partial A\right) \\
&= \operatorname{tr}\left(A^{-1}\partial A\right) \\
\implies & \boxed{\partial \det A = \det A \operatorname{tr}\left(A^{-1}\partial A\right)} \\
&= -\det A \operatorname{tr}\left(A(-A^{-2})\partial A\right) \\
& \boxed{\partial \det A = -\det A \operatorname{tr}\left(A\partial(A^{-1})\right)}
\end{aligned} \tag{A.4}$$

A.3 Metric Density

Let $g \equiv \det[g_{\mu\nu}]$ in this context. We use our result from (A.4). For variational derivatives w.r.t. to the inverse metric,

$$\begin{aligned}
&\boxed{\delta g = -g g_{\mu\nu}\delta g^{\mu\nu}} \\
&\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}} \times (-\delta g) \\
&\boxed{\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}}
\end{aligned} \tag{A.5}$$

For partial derivatives,

$$\boxed{\partial_i g = g g^{ab} \partial_i g_{ba}}$$

$$\partial_i \sqrt{-g} = \frac{1}{2\sqrt{-g}} \partial_i(-g) \tag{A.6}$$

$$\boxed{\partial_i \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{ab} \partial_i g_{ab}}$$

A.4 Connection Coefficients

$$\Gamma^i_{ij} = \frac{1}{2} g^{ia} (\cancel{\partial_i g_{aj}} + \partial_j g_{ia} - \cancel{\partial_a g_{ij}}) \tag{A.7}$$

$$= \frac{1}{2} g^{ia} \partial_j g_{ia} \tag{A.8}$$

$$\boxed{\Gamma^i_{ij} = \frac{1}{2g} \partial_j g} \tag{A.9}$$

$$\text{or } \boxed{\Gamma^i_{ij} = \frac{1}{\sqrt{-g}} \partial_j \sqrt{-g}} \tag{A.10}$$

$$g^{jk} \Gamma^i_{jk} = \frac{1}{2} g^{jk} g^{ia} (\partial_j g_{ak} + \partial_k g_{ja} - \partial_a g_{jk}) \tag{A.11}$$

$$= g^{jk} g^{ia} \partial_j g_{ka} - \frac{1}{2} g^{jk} g^{ia} \partial_a g_{jk} \tag{A.12}$$

$$= \cancel{g^{jk} \partial_j (g^{ia} g_{ka})} - g^{jk} \partial_j g^{ia} g_{ka} - \frac{1}{2} g^{ia} g^{jk} \partial_a g_{jk} \tag{A.13}$$

$$= -\frac{1}{\sqrt{-g}} \sqrt{-g} \partial_a g^{ia} - \frac{1}{\sqrt{-g}} g^{ia} \partial_a \sqrt{-g} \tag{A.14}$$

$$\boxed{g^{jk} \Gamma^i_{jk} = -\frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g} g^{ia}\right)} \tag{A.15}$$

A.5 Covariant Derivatives

$$\nabla_i v^i = \partial_i v^i + \Gamma^i_{ij} v^j \tag{A.16}$$

$$= \frac{1}{\sqrt{-g}} \sqrt{-g} \partial_i v^i + \frac{1}{\sqrt{-g}} \partial_j \sqrt{-g} v^j \tag{A.17}$$

$$\boxed{\nabla_i v^i = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} v^i)} \tag{A.18}$$

$$\Longrightarrow \boxed{\nabla_i \nabla^i \phi = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} \partial^i \phi)} \tag{A.19}$$

As a consistency check, we do the divergence of a covector field:

$$g^{ij} \nabla_i \omega_j = g^{ij} \partial_i \omega_j - g^{ij} \Gamma^k_{ij} \omega_k \tag{A.20}$$

$$= \frac{1}{\sqrt{-g}} \sqrt{-g} g^{ij} \partial_i \omega_j - \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ik}) \omega_k \tag{A.21}$$

$$\boxed{\nabla^i \omega_i = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \omega_j)} \tag{A.22}$$

which agrees with (A.18)

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