Structure of black holes in theories beyond general relativity
LIGO Progress Report 1

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1 Introduction

1.1 Background and Motivation

Because we know general relativity is not complete, as outlined earlier in the proposal, we are on the search for a fuller theory of gravity. From an effective field theory standpoint, UV complete theories of gravity manifest as higher-order corrections to GR in the low energy limit. Therefore, there is a hope that LIGO and similar experiments can detect variations from the long-standing general theory of relativity in the strong field limit. Ultimately, in order to see the difference between pure GR and any generic correction, we need simulate compact binary merger to determine what the characteristics effects of the correction in experiment.

1.2 Kerr eigenfunctions

Before embarking on merger simulations, we focus on stationary and isolated, black holes, where no matter is unmodeled. The program we are developing from ground up should be generalizable to binary black holes and systems with modelled matter.

The Kerr solution will be the background for a family of solutions to non-minimal coupling terms as a perturbation to the Kerr solution under the Einstein-Hilbert action. The perturbation parameter ϵ will parameterize the solutions of these "bumpy black holes". The bumps can arise from corrections to general relativity or matter itself. What is important is that we are attempting to develop a formalism where we find eigenfunctions for the linearized Einstein tensor. This way, the analysis of bumpy black holes can be greatly simplified and clarified.

2 Progress

I derived all the following results from the ground up in the first two weeks.

2.1 Perturbation Theory

We have a 1-parameter family of geometries, described by

$$g_{ab}{}^{(\lambda)} = g_{ab}{}^{(0)} + \lambda \left.\frac{dg_{ab}{}^{(\lambda)}}{d\lambda}\right|_{\lambda=0} + \mathcal{O}(\lambda^2) \tag{1}$$

$$\equiv g_{ab}{}^{(0)} + \lambda h_{ab} + \mathcal{O}(\lambda^2) \tag{2}$$

For our purposes at the moment, $g_{ab}{}^{(0)}$ is the background Ricci-flat spacetime (corresponding to Schwarzschild or Kerr), and h_{ab} is our metric perturbation.

2.2 Connection on a Background

We have the difference of connections, where $\nabla_a^{(\lambda)}$ is compatible with the metric $g_{bc}{}^{(\lambda)}$:

$$(\nabla_a^{(\lambda)} - \nabla_a^{(0)})v^b = C^b_{ac}v^c \tag{3}$$

$$(\nabla_a^{(\lambda)} - \nabla_a^{(0)})\omega_b = -C^c_{ab}\omega_c \tag{4}$$

where C^c_{ab} is a function of λ.

Therefore, we have from $0 = \nabla_c^{(\lambda)} g_{ab}{}^{(\lambda)}$, we have two identities:

$$C^c_{ab} = \frac{1}{2}g^{cd}{}_{(\lambda)}\left(\nabla_a{}^{(0)}g_{db}{}^{(\lambda)} + \nabla_b{}^{(0)}g_{ad}{}^{(\lambda)} - \nabla_d{}^{(0)}g_{ab}{}^{(\lambda)}\right) \tag{5}$$

$$C^c_{ab} = \frac{1}{2}g^{cd}{}_{(\lambda)}\left(\partial_a g_{db}{}^{(\lambda)} + \partial_b g_{ad}{}^{(\lambda)} - \partial_d g_{ab}{}^{(\lambda)}\right) - \frac{1}{2}g^{cd}{}_{(0)}\left(\partial_a g_{db}{}^{(0)} + \partial_b g_{ad}{}^{(0)} - \partial_d g_{ab}{}^{(0)}\right) \tag{6}$$

For notational convenience let $\tilde{\nabla}_a \equiv {\nabla_a}^{(0)}$ and $\nabla_a \equiv {\nabla_a}^{(\lambda)}$. The Riemann curvature tensor is

$$
\begin{aligned}
{R_{abc}}^d \omega_d &\equiv [\nabla_a, \nabla_b]\omega_c && (7)\\
&= \nabla_a \nabla_b \omega_c - (a \leftrightarrow b) && (8)\\
&= \tilde{\nabla}_a(\nabla_b \omega_c) - \cancel{C^d_{ab}(\nabla_d \omega_c)} - C^d_{ac}(\nabla_b \omega_d) - (a \leftrightarrow b) && (9)\\
&= \tilde{\nabla}_a(\tilde{\nabla}_b \omega_c - C^d_{bc}\omega_d) - C^d_{ac}(\tilde{\nabla}_b \omega_d - C^e_{bd}\omega_e) - (a \leftrightarrow b) && (10)\\
&= \tilde{\nabla}_a \tilde{\nabla}_b \omega_c - \tilde{\nabla}_a C^d_{bc}\omega_d \cancel{- C^d_{bc}\tilde{\nabla}_a \omega_d - C^d_{ac}\tilde{\nabla}_b \omega_d} + C^d_{ac}C^e_{bd}\omega_e - (a \leftrightarrow b) && (11)\\
&= \tilde{\nabla}_{[a}\tilde{\nabla}_{b]}\omega_c - \tilde{\nabla}_{[a}C^d_{b]c}\omega_d + C^d_{c[a}C^e_{b]d}\omega_e && (12)\\
&= \left({R^d}_{abc}{}^{(0)} - \tilde{\nabla}_{[a}C^d_{b]c} + C^e_{c[a}C^d_{b]e}\right)\omega_d && (13)\\
\implies \boxed{{R_{abc}}^d = {R_{abc}}^d{}_{(0)} - \tilde{\nabla}_{[a}C^d_{b]c} + C^e_{c[a}C^d_{b]e}} & && (14)
\end{aligned}
$$

2.3 Linearized Einstein Operator (possibly Lichnerowicz)

Let $\tilde{\nabla}_a \equiv {\nabla_a}^{(0)}$ and $g_{ab} = {g_{ab}}^{(\lambda)}$ unless otherwise specified.

$$
\begin{aligned}
C^c_{ab} &= \frac{1}{2} g^{cd}\left(\tilde{\nabla}_a g_{db} + \tilde{\nabla}_b g_{ad} - \tilde{\nabla}_d g_{ab}\right) && (15)\\
{C^c}_{ab}{}^{(0)} &= 0 && (16)\\
\implies C^c_{ab} &= \mathcal{O}(\lambda) && (17)\\
C^c_{ab} &= \frac{1}{2}\lambda g^{cd(0)}\left(\tilde{\nabla}_a h_{db} + \tilde{\nabla}_b h_{ad} - \tilde{\nabla}_d h_{ab}\right) + \mathcal{O}(\lambda^2) && (18)
\end{aligned}
$$

We have

$$
\begin{aligned}
{R_{abc}}^d &= {R_{abc}}^d{}_{(0)} - \tilde{\nabla}_{[a}C^d_{b]c} + \mathcal{O}(\lambda^2) && (19)\\
\implies R_{ac} &= {R_{ac}}^{(0)} - \tilde{\nabla}_{[a}C^d_{d]c} + \mathcal{O}(\lambda^2) && (20)\\
&= {R_{ac}}^{(0)} - \frac{1}{2}\lambda g^{de(0)}\left(\tilde{\nabla}_a\tilde{\nabla}_d h_{ec} + \tilde{\nabla}_a\tilde{\nabla}_c h_{de} - \tilde{\nabla}_a\tilde{\nabla}_e h_{dc} - (a \leftrightarrow d)\right) + \mathcal{O}(\lambda^2) && (21)\\
&= {R_{ac}}^{(0)} - \frac{1}{2}\lambda\left([\tilde{\nabla}_a, \tilde{\nabla}^e]h_{ec} + \tilde{\nabla}_a\tilde{\nabla}_c h - \tilde{\nabla}^e\tilde{\nabla}_c h_{ae} - \tilde{\nabla}_a\tilde{\nabla}^d h_{dc} + \tilde{\nabla}_d\tilde{\nabla}^d h_{ac}\right) + \mathcal{O}(\lambda^2) && (22)\\
\boxed{R_{ac} = {R_{ac}}^{(0)} - \frac{1}{2}\lambda\left(\tilde{\nabla}_a\tilde{\nabla}_c h - \tilde{\nabla}^e\tilde{\nabla}_{(a}h_{c)e} + \tilde{\nabla}_d\tilde{\nabla}^d h_{ac}\right) + \mathcal{O}(\lambda^2)} & && (23)
\end{aligned}
$$

where we have h_{ab} raised and lowered (and traced) by the background metric $g^{cd(0)}$.

Furthermore, we have

$$
\begin{aligned}
R &= g^{ac}R_{ac} && (24)\\
&= (g^{ac(0)} - \lambda h^{ac}){R_{ac}}^{(0)} - \frac{1}{2}\lambda g^{ac(0)}g^{de(0)}\left(\tilde{\nabla}_a\tilde{\nabla}_c h - \tilde{\nabla}^e\tilde{\nabla}_{(a}h_{c)e} + \tilde{\nabla}_d\tilde{\nabla}^d h_{ac}\right) + \mathcal{O}(\lambda^2) && (25)\\
&= R^{(0)} - \lambda h^{ac}{R_{ac}}^{(0)} - \frac{1}{2}\lambda\left(\tilde{\nabla}_a\tilde{\nabla}^a h - 2\tilde{\nabla}^e\tilde{\nabla}^a h_{ae} + \tilde{\nabla}_d\tilde{\nabla}^d h\right) + \mathcal{O}(\lambda^2) && (26)\\
\boxed{R = R^{(0)} - \lambda\left(h^{ac}{R_{ac}}^{(0)} + \tilde{\nabla}_d\tilde{\nabla}^d h - \tilde{\nabla}^c\tilde{\nabla}^d h_{cd}\right) + \mathcal{O}(\lambda^2)} & && (27)
\end{aligned}
$$

Therefore the linearized Einstein tensor is

$$
\begin{aligned}
G_{ab} \equiv& R_{ab} - \frac{1}{2} R g_{ab} && (28)\\
=& {R_{ab}}^{(0)} - \frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_{(a} h_{b)e} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ab}\right) && (29)\\
&- \frac{1}{2}({g_{ab}}^{(0)} + \lambda h_{ab}) \left[R^{(0)} - \lambda \left(h^{cd} {R_{cd}}^{(0)} + \tilde{\nabla}_d \tilde{\nabla}^d h - \tilde{\nabla}^c \tilde{\nabla}^d h_{cd}\right)\right] + \mathcal{O}(\lambda^2) && (30)\\
=& {G_{ab}}^{(0)} - \frac{1}{2}\lambda h_{ab} R^{(0)} + \frac{1}{2}\lambda {g_{ab}}^{(0)} h^{cd} {R_{cd}}^{(0)} && (31)\\
&- \frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_{(a} h_{b)e} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ab}\right) + \frac{1}{2}\lambda {g_{ab}}^{(0)} \left(\tilde{\nabla}_d \tilde{\nabla}^d h - \tilde{\nabla}^c \tilde{\nabla}^d h_{cd}\right) + \mathcal{O}(\lambda^2) && (32)
\end{aligned}
$$

If we have a Ricci-flat background, ${R_{cd}}^{(0)} = 0$,

$$
\boxed{G_{ab} = -\frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_{(a} h_{b)e} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ab} - {g_{ab}}^{(0)} \tilde{\nabla}_d \tilde{\nabla}^d h + {g_{ab}}^{(0)} \tilde{\nabla}^c \tilde{\nabla}^d h_{cd}\right) + \mathcal{O}(\lambda^2)} \qquad (33)
$$

which agrees with the Fierz-Pauli equation for massless spin-2 bosons in a Minkowski background.

We can also note that $\lambda \nabla_a = \lambda \tilde{\nabla}_a + \mathcal{O}(\lambda^2)$, so

$$
G_{ab} = -\frac{1}{2}\lambda \left(\nabla_a \nabla_b h - \nabla^e \nabla_{(a} h_{b)e} + \nabla_d \nabla^d h_{ab} - g_{ab} \nabla_d \nabla^d h + g_{ab} \nabla^c \nabla^d h_{cd}\right) + \mathcal{O}(\lambda^2) \qquad (34)
$$

2.4 Gauge conditions

2.4.1 Covariant Derivative Commutator derivation

Given that $[\tilde{\nabla}_a, \tilde{\nabla}_b]\omega_c = -{R^d}_{cab}^{(0)} \omega_d$, we have

$$
\begin{aligned}
[\tilde{\nabla}_a, \tilde{\nabla}_b](h_{cd} v^d) &= -{R^e}_{cab}^{(0)} (h_{ed} v^d) && (35)\\
\tilde{\nabla}_a \tilde{\nabla}_b h_{cd} v^d + \cancel{\tilde{\nabla}_b h_{cd} \tilde{\nabla}_a v^d + \tilde{\nabla}_a h_{cd} \tilde{\nabla}_b v^d} + h_{cd} \tilde{\nabla}_a \tilde{\nabla}_b v^d - (a \leftrightarrow b) &= -{R^e}_{cab}^{(0)} (h_{ed} v^d) && (36)\\
[\tilde{\nabla}_a, \tilde{\nabla}_b] h_{cd} v^d + h_{ce} [\tilde{\nabla}_a, \tilde{\nabla}_b] v^e &= -{R^e}_{cab}^{(0)} (h_{ed} v^d) && (37)\\
[\tilde{\nabla}_a, \tilde{\nabla}_b] h_{cd} v^d + h_{ce} {R^e}_{dab}^{(0)} v^d &= -{R^e}_{cab}^{(0)} h_{ed} v^d && (38)\\
\boxed{[\tilde{\nabla}_a, \tilde{\nabla}_b] h_{cd} = -{R^e}_{cab}^{(0)} h_{ed} - {R^e}_{dab}^{(0)} h_{ce}} & && (39)
\end{aligned}
$$

2.4.2 Lorenz Gauge of the Trace-reverse of Metric Perturbation

In Lorenz gauge, $0 = \tilde{\nabla}^a \bar{h}_{ab} = \tilde{\nabla}^a h_{ab} - \frac{1}{2} g_{ab} \tilde{\nabla}^a h$ in $3+1$ dimensions with a Ricci-flat background

$$
\begin{aligned}
G_{ab} &= -\frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_{(a} h_{b)e} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ab} - g_{ab} \tilde{\nabla}_d \tilde{\nabla}^d h + g_{ab} \tilde{\nabla}^c \tilde{\nabla}^d h_{cd} \right) + \mathcal{O}(\lambda^2) && (40) \\
&= -\frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_{(a} h_{b)e} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ab} - g_{ab} \tilde{\nabla}_d \tilde{\nabla}^d h + \frac{1}{2} g_{ab} \tilde{\nabla}^c (g_{cd} \tilde{\nabla}^d h) \right) + \mathcal{O}(\lambda^2) && (41) \\
&= -\frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_{(a} h_{b)e} + \tilde{\nabla}_d \tilde{\nabla}^d h_{ab} - \frac{1}{2} g_{ab} \tilde{\nabla}_d \tilde{\nabla}^d h \right) + \mathcal{O}(\lambda^2) && (42) \\
&= -\frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_a h_{be} - \tilde{\nabla}^e \tilde{\nabla}_b h_{ae} + \tilde{\nabla}_d \tilde{\nabla}^d \bar{h}_{ab} \right) + \mathcal{O}(\lambda^2) && (43) \\
&= -\frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_a \left(\bar{h}_{be} + \frac{1}{2} g_{be} h \right) - \tilde{\nabla}^e \tilde{\nabla}_b \left(\bar{h}_{ae} + \frac{1}{2} g_{ae} h \right) + \tilde{\nabla}_d \tilde{\nabla}^d \bar{h}_{ab} \right) + \mathcal{O}(\lambda^2) && (44) \\
&= -\frac{1}{2}\lambda \left(\tilde{\nabla}_a \tilde{\nabla}_b h - \tilde{\nabla}^e \tilde{\nabla}_a \bar{h}_{be} - \frac{1}{2} \tilde{\nabla}_b \tilde{\nabla}_a h - \tilde{\nabla}^e \tilde{\nabla}_b \bar{h}_{ae} - \frac{1}{2} \tilde{\nabla}_a \tilde{\nabla}_b h + \tilde{\nabla}_d \tilde{\nabla}^d \bar{h}_{ab} \right) + \mathcal{O}(\lambda^2) && (45) \\
&= -\frac{1}{2}\lambda \left(-\tilde{\nabla}^e \tilde{\nabla}_{(a} \bar{h}_{b)e} + \tilde{\nabla}_d \tilde{\nabla}^d \bar{h}_{ab} \right) + \mathcal{O}(\lambda^2) && (46) \\
&= -\frac{1}{2}\lambda \left(-g^{ec} \left([\tilde{\nabla}_c, \tilde{\nabla}_a] \bar{h}_{be} + (a \leftrightarrow b) \right) + \tilde{\Box} \bar{h}_{ab} \right) + \mathcal{O}(\lambda^2) && (47) \\
&= -\frac{1}{2}\lambda \left(-g^{ec} \left(-R^{d}{}_{bca}{}^{(0)} \bar{h}_{de} - R^{d}{}_{eca}{}^{(0)} \bar{h}_{bd} + (a \leftrightarrow b) \right) + \tilde{\Box} \bar{h}_{ab} \right) + \mathcal{O}(\lambda^2) && (48) \\
&= -\frac{1}{2}\lambda \left(+ \left(R^{d}{}_{b}{}^{e}{}_{a}{}^{(0)} \bar{h}_{de} + \cancel{R^{d}{}_{e}{}^{e}{}_{a}{}^{(0)} \bar{h}_{bd}} + (a \leftrightarrow b) \right) + \tilde{\Box} \bar{h}_{ab} \right) + \mathcal{O}(\lambda^2) && (49) \\
&\boxed{G_{ab} = -\frac{1}{2}\lambda \left(2R^{c}{}_{a}{}^{d}{}_{b}{}^{(0)} \bar{h}_{cd} + \tilde{\Box} \bar{h}_{ab} \right) + \mathcal{O}(\lambda^2)} && (50)
\end{aligned}
$$

2.4.3 Infinitesimal Gauge Transformation

We see that infinitesimal diffeomorphism $x^a \mapsto x'^{a'} = x^{a'} + \kappa^{a'}$, is equivalent to an infinitesimal gauge transformation of the metric at linear order:

$$
\begin{aligned}
g^{ab}(x) &\mapsto g^{a'b'}(x') && (51) \\
&= \frac{\partial x'^{a'}}{\partial x^a} \frac{\partial x'^{b'}}{\partial x^b} g^{ab}(x) && (52) \\
&= (\delta_a^{a'} + \partial_a \kappa^{a'})(\delta_b^{b'} + \partial_b \kappa^{b'}) g^{ab}(x) && (53) \\
&= \left(\delta_a^{a'} \delta_b^{b'} + \delta_a^{a'} \partial_b \kappa^{b'} + \partial_a \kappa^{a'} \delta_b^{b'} + \mathcal{O}(\kappa^2) \right) g^{ab}(x) && (54) \\
&= g^{a'b'}(x) + \partial^{a'} \kappa^{b'} + \partial^{b'} \kappa^{a'} + \mathcal{O}(\kappa^2) && (55)
\end{aligned}
$$

Therefore for first order perturbations, $h_{ab} \mapsto h_{ab} + \nabla_a^{(0)} \kappa_b + \nabla_b^{(0)} \kappa_a$ is a gauge transformation for arbitrary infinitesimal covector field κ_a.

2.5 Decoupling Limit of dCS

The action is

$$I = \int d^4x\sqrt{-g}\left[\frac{m_p^2}{2}R - \frac{1}{2}\partial_a\theta\partial^a\theta + \epsilon\mathcal{L}_{\text{int}}\right] \tag{56}$$

Imposing the principle of stationary action,

$$
\begin{aligned}
0 &= \delta I && (57)\\
&= \int\left\{\delta\sqrt{-g}\left[\frac{m_p^2}{2}R - \frac{1}{2}\partial_a\theta\partial^a\theta + \epsilon\mathcal{L}_{\text{int}}\right] + \sqrt{-g}\delta\left[\frac{m_p^2}{2}R - \frac{1}{2}\partial_a\theta\partial^a\theta + \epsilon\mathcal{L}_{\text{int}}\right]\right\}d^4x && (58)\\
&= \int d^4x\sqrt{-g}\left\{-\frac{1}{2}g_{ab}\delta g^{ab}\left[\frac{m_p^2}{2}R - \frac{1}{2}\partial_c\theta\partial^c\theta + \epsilon\mathcal{L}_{\text{int}}\right] + \frac{m_p^2}{2}\delta R + \delta\left[-\frac{1}{2}\partial_c\theta\partial^c\theta + \epsilon\mathcal{L}_{\text{int}}\right]\right\} && (59)\\
&= \int d^4x\sqrt{-g}\left\{-\frac{1}{2}g_{ab}\delta g^{ab}\left[\frac{m_p^2}{2}R - \frac{1}{2}\partial_c\theta\partial^c\theta + \epsilon\mathcal{L}_{\text{int}}\right] + \frac{m_p^2}{2}R_{ab}\delta g^{ab} - \frac{1}{2}\delta(\partial_c\theta\partial^c\theta) + \delta\left[\epsilon\mathcal{L}_{\text{int}}\right]\right\} && (60)\\
&= \int d^4x\sqrt{-g}\delta g^{ab}\left\{\frac{m_p^2}{2}\left(R_{ab} - \frac{1}{2}g_{ab}R\right) - \frac{1}{2}g_{ab}\left[-\frac{1}{2}\partial_c\theta\partial^c\theta + \epsilon\mathcal{L}_{\text{int}}\right]\right. && (61)\\
&\quad \left.-\frac{1}{2}\frac{\delta}{\delta g^{ab}}(g^{cd}\partial_c\theta\partial_d\theta) + \frac{\delta}{\delta g^{ab}}(\epsilon\mathcal{L}_{\text{int}})\right\} + \int d^4x\sqrt{-g}\left\{-\frac{\frac{1}{2}\delta(\partial_c\theta\partial^c\theta)}{\delta\theta}\delta\theta + \epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\delta\theta\right\} && (62)\\
&= \int d^4x\sqrt{-g}\delta g^{ab}\left\{\frac{m_p^2}{2}G_{ab} - \frac{1}{2}g_{ab}\left[-\frac{1}{2}\partial_c\theta\partial^c\theta + \epsilon\mathcal{L}_{\text{int}}\right] - \frac{1}{2}\delta_a^c\delta_b^d\partial_c\theta\partial_d\theta + \epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right\} && (63)\\
&\quad + \int d^4x\sqrt{-g}\left\{-\frac{\partial_c\theta\partial^c\delta\theta}{\delta\theta}\delta\theta + \epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\delta\theta\right\} && (64)\\
&= \frac{1}{2}\int d^4x\sqrt{-g}\delta g^{ab}\left\{m_p^2G_{ab} - \left[\partial_a\theta\partial_b\theta - \frac{1}{2}g_{ab}\partial_c\theta\partial^c\theta\right] + \epsilon\mathcal{L}_{\text{int}}g_{ab} + 2\epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right\} && (65)\\
&\quad + \int d^4x\delta\theta\left\{+\partial^c(\sqrt{-g}\nabla_c\theta) + \epsilon\sqrt{-g}\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right\} && (66)\\
0 &= \frac{1}{2}\int d^4x\sqrt{-g}\delta g^{ab}\left\{m_p^2G_{ab} - \left[\partial_a\theta\partial_b\theta - \frac{1}{2}g_{ab}\partial_c\theta\partial^c\theta\right] + \epsilon\mathcal{L}_{\text{int}}g_{ab} + 2\epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right\} && (67)\\
&\quad + \int d^4x\sqrt{-g}\delta\theta\left\{+\nabla^c\nabla_c\theta + \epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right\} && (68)
\end{aligned}
$$

Therefore our equations of motion are:

$$m_p^2G_{ab} + \underbrace{\epsilon\mathcal{L}_{\text{int}}g_{ab} + 2\epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}}_{\epsilon C_{ab}} = \underbrace{\partial_a\theta\partial_b\theta - \frac{1}{2}g_{ab}\partial_c\theta\partial^c\theta}_{T_{ab}{}^{(\theta)}} \tag{69}$$

$$\Box\theta = \underbrace{-\epsilon\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}}_{S} \tag{70}$$

We have the perturbative expansion from a Ricci-flat, scalarless background:

$$\theta = 0 + \epsilon\theta^{(1)} + \frac{1}{2}\epsilon^2\theta^{(2)} + \mathcal{O}(\epsilon^3) \tag{71}$$

$$g_{ab} = g_{ab}{}^{(0)} + \epsilon h_{ab}{}^{(1)} + \frac{1}{2}\epsilon^2 h_{ab}{}^{(2)} + \mathcal{O}(\epsilon^3) \tag{72}$$

$$T_{ab}{}^{(\theta)} = \mathcal{O}(\epsilon^2) \tag{73}$$

$$R_{abcd} = \mathcal{O}(1) \tag{74}$$

$$\mathcal{L}_{\text{int}} = \mathcal{O}(\epsilon) \tag{75}$$

$$S = \mathcal{O}(\epsilon) \tag{76}$$

$$\epsilon C_{ab} = \mathcal{O}(\epsilon^2) \tag{77}$$

$$G_{ab} = -\frac{1}{2}\epsilon\left(2R^{c\ \ d\ (0)}_{\ a\ \ b}\bar{h}_{cd}{}^{(1)} + \Box^{(0)}\bar{h}_{ab}{}^{(1)}\right) + \mathcal{O}(\epsilon^2) \tag{78}$$

So in the decoupling limit of $\epsilon \to 0$,

2.5.1 Zeroth Order

Just the Kerr solution with no scalar.

2.5.2 First Order

$$\Box^{(0)}\left(\epsilon\theta^{(1)}\right) = -\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right)^{(0)} \tag{79}$$

$$\Box^{(0)}\theta^{(1)} = -\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right)^{(0)} \tag{80}$$

and

$$m_p^2 G_{ab}^{(1)} + \epsilon\mathcal{L}_{\text{int}}^{(0)} g_{ab}^{(0)} + 2\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right)^{(0)} = 0 \tag{81}$$

$$m_p^2 G_{ab}^{(1)} = 0 \tag{82}$$

$$\left(2R^{c\ \ d\ (0)}_{\ a\ \ b} + \delta^c_a\delta^d_b\Box^{(0)}\right)\bar{h}_{cd}{}^{(1)} = 0 \tag{83}$$

where a solution is $\bar{h}_{cd}{}^{(1)} = 0$.

2.5.3 Second Order

Now at $\mathcal{O}(\epsilon^2)$ order, assuming $\bar{h}_{cd}{}^{(1)} = 0$,

$$m_p^2 G_{ab}^{(2)} + \epsilon\mathcal{L}_{\text{int}}^{(1)} g_{ab}^{(0)} + 2\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right)^{(1)} = \partial_a\left(\epsilon\theta^{(1)}\right)\partial_b\left(\epsilon\theta^{(1)}\right) - \frac{1}{2}g_{ab}^{(0)}\partial_c\left(\epsilon\theta^{(1)}\right)\partial^c\left(\epsilon\theta^{(1)}\right) \tag{84}$$

which reduces to

$$G_{ab}^{(2)} = \underbrace{m_p^{-2}\left[\underbrace{-\epsilon\mathcal{L}_{\text{int}}^{(1)} g_{ab}^{(0)} - 2\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right)^{(1)}}_{-\epsilon C_{ab}^{(1)}} + \underbrace{\epsilon^2\partial_a\theta^{(1)}\partial_b\theta^{(1)} - \frac{1}{2}\epsilon^2 g_{ab}^{(0)}\partial_c\theta^{(1)}\partial^c\theta^{(1)}}_{T_{ab}^{(2)}}\right]}_{S_{ab}^{(2)}} \tag{85}$$

$$\implies -\frac{1}{4}\left(2R^{c\ \ d\ (0)}_{\ a\ \ b} + \delta^c_a\delta^d_b\Box^{(0)}\right)\bar{h}_{cd}{}^{(2)} = S_{ab}^{(2)} \tag{86}$$

2.5.4 Third Order

We need to find θ to subsecond order in ϵ:

$$\square^{(0)}\left(\frac{1}{2}\epsilon^2\theta^{(2)}\right) = -\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right)^{(2)} \tag{87}$$

$$\square^{(0)}\theta^{(2)} = -\frac{2}{\epsilon}\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta\theta}\right)^{(2)} \tag{88}$$

Then we have to $\mathcal{O}(\epsilon^3)$ order, assuming $\bar{h}_{cd}{}^{(1)} = 0$,

$$G_{ab}^{(3)} = \underbrace{m_p^{-2}\left[\underbrace{-\epsilon\mathcal{L}_{\text{int}}^{(2)}g_{ab}^{(0)} - 2\epsilon\left(\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{ab}}\right)^{(2)}}_{-\epsilon C_{ab}^{(2)}} + \underbrace{\frac{1}{2}\epsilon^2\left(\partial_a\theta^{(1)}\partial_b\theta^{(2)} + \partial_a\theta^{(2)}\partial_b\theta^{(1)} - g_{ab}^{(0)}\partial_c\theta^{(1)}\partial^c\theta^{(2)}\right)}_{T_{ab}^{(3)}}\right]}_{S_{ab}^{(3)}} \tag{89}$$

$$\implies -\frac{1}{12}\left(2R^{c\ \ d}_{\ a\ b}{}^{(0)} + \delta_a^c\delta_b^d\square^{(0)}\right)\bar{h}_{cd}{}^{(3)} = S_{ab}^{(3)} \tag{90}$$

2.6 Birkhoff's Theorem

2.6.1 Spherical Symmetry

Assuming a spherically symmetric $3+1$ dimensional spacetime, we can choose coordinates so that the metric has the general form:

$$\mathrm{d}s^2 = A(t,r)\mathrm{d}t^2 + B(t,r)\mathrm{d}t\,\mathrm{d}r + C(t,r)\mathrm{d}r^2 + D(t,r)\mathrm{d}\Omega^2 \tag{91}$$

We can transform our coordinates (t,r) so that r becomes $\sqrt{D}$. We choose the positive root because we want the angular coordinates to have positive Lorentzian signature (If we choose the negative convention our final metric change to reflect the convention change). Therefore we can always rewrite our spherically symmetric metric as

$$\mathrm{d}s^2 = A(t,r)\mathrm{d}t^2 + B(t,r)\mathrm{d}t\,\mathrm{d}r + C(t,r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{92}$$

where we have chosen the coordinate r specifically to give the spatial 2-sphere an r^2 areal dependence in the 4-fold.

Given any $A(t,r), B(t,r), C(t,r)$, we can transform the t coordinates so that our new coordinates, $t'(t,r)$ and r, gives

$$\mathrm{d}t'^2 = \left(\frac{\partial t'}{\partial t}\mathrm{d}t + \frac{\partial t'}{\partial r}\mathrm{d}r\right)^2 \tag{93}$$

$$\mathrm{d}t'^2 = \left(\frac{\partial t'}{\partial t}\right)^2\mathrm{d}t^2 + 2\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}\mathrm{d}t\mathrm{d}r + \left(\frac{\partial t'}{\partial r}\right)^2\mathrm{d}r^2 \tag{94}$$

$$D(t',r)\left(\frac{\partial t'}{\partial t}\right)^2 = A(t,r) \tag{95}$$

$$D(t',r)\left(2\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}\right)^2 = B(t,r) \tag{96}$$

$$E(t',r) - D(t',r)\left(\frac{\partial t'}{\partial r}\right)^2 = C(t,r) \tag{97}$$

Since we have three equations for three variables $t'(t,r), D\left(t'(t,r),r\right), E\left(t'(t,r),r\right)$, the equations are always soluble up given initial conditions. The choice of initial conditions is part of the gauge choice of our coordinate system. Then the line element is

$$\mathrm{d}s^2 = D(t,r)\mathrm{d}t^2 + E(t,r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{98}$$

We see that we have two functional degrees of freedom assuming spherical symmetry. Once the vacuum Einstein Field Equations are imposed, we will see that only a real valued parameter will remain as a degree of freedom.

2.6.2 Vacuum Einstein Field Equations

In regions where D and E do not blow up or go to 0, we can renaming our metric degrees of freedom, in two steps:

$$\mathrm{d}s^2 = -e^{2\psi(t,r)}f(t,r)\mathrm{d}t^2 + \frac{1}{f(t,r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{99}$$

$$\mathrm{d}s^2 = -e^{2\psi(t,r)}\left(1 - \frac{2m(t,r)}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2m(t,r)}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{100}$$

In complete vacuum ${T^\mu}_\nu = 0$, we have that for the Einstein tensor ${G^\mu}_\nu$ with the help of Mathematica,

$$0 = {G^t}_t = \frac{-2\partial_r m(t,r)}{r^2} \tag{101}$$

$$0 = {G^r}_t = \frac{2\partial_t m(t,r)}{r^2} \tag{102}$$

$$0 = {G^r}_r - {G^t}_t = \frac{2}{r}\left(1 - \frac{2m(t,r)}{r}\right)\partial_r\psi(t,r) \tag{103}$$

By equation (101), $m(t,r) = m(t)$ and by equation (102), $m(t,r) = m(r)$. Therefore $m(t,r)$ is a real constant.
Now by equation (103), we have $\psi(t,r) = \psi(t)$.
We can then rescale $t \mapsto e^{-\psi(t)}t$, so that $g_{tt} = -\left(1 - \frac{2m}{r}\right)$ and all other metric components stay the same.

Therefore the unique spherically symmetric solution to the vacuum Einstein Field equations with $\Lambda = 0$ is the Schwarzschild solution:

$$\boxed{\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2} \tag{104}$$

for some coordinates with the $- + ++$ Lorentzian signature.

Notice we see that any spherically symmetric solution must be asymptotically flat (as $r \to \infty$) and static (with respect to the time-like vector $\frac{\partial}{\partial t}$); we did not impose these conditions.

Therefore, there is no gravitational monopole radiation.

2.6.3 Komar Mass

It turns out the Komar mass integral of the Schwarzschild solution is m, so m really does correspond to a physical mass of the metric.

2.7 Weyl-Lewis-Papapetrou

Given a time-like (t direction) and an "azimuthal" space-like (ϕ direction) Killing vector fields, we ostensibly have a metric

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{ij}dx^i dx^j \tag{105}$$

for $i, j \in \{2, 3\}$.

Due to Papapetrou, we make a gauge transformation so that any stationary, axisymmetric spacetime is:

$$\mathrm{d}s^2 = -V(\mathrm{d}t - w\mathrm{d}\phi)^2 + V^{-1}\rho^2\mathrm{d}\phi^2 + \Omega^2(\mathrm{d}\rho^2 + \Lambda\mathrm{d}z^2) \tag{106}$$

where $V(\rho,z), w(\rho,z), \Omega(\rho,z), \Lambda(\rho,z)$ are the four functional degrees of freedom.

In Ricci-flat spacetimes, this reduces to three functional degrees of freedom.

3 Challenges

Some challenges and near term goals are

- Prove Froebenius' theorem and, in turn, prove the Weyl-Lewis-Papapetrou (WLP) form for general stationary axisymmetric spacetimes.

- Implement the perturbations off the Schwarzschild solution in WLP formalism. I need to figure out how to do this explicitly in xAct/xCoba. Then I should proceed with the Kerr background after this is developed.
- Verify these solutions correspond to nice gauge conditions. We want our solutions space to be a well defined number of dimensions.
- Determine definitively whether the Einstein operator is an elliptic operator.

See proposal for long term goals.

Appendices

A Miscellaneous Identities Used in Proofs

A.1 Metric

$$\begin{aligned}
\frac{\partial g^{\rho\sigma}}{\partial g_{\mu\nu}} &= \frac{\partial(g^{\rho\rho'}g^{\sigma\sigma'}g_{\rho'\sigma'})}{\partial g_{\mu\nu}} = \frac{\partial g^{\rho\rho'}}{\partial g_{\mu\nu}}g^{\sigma\sigma'}g_{\rho'\sigma'} + g^{\rho\rho'}\frac{\partial g^{\sigma\sigma'}}{\partial g_{\mu\nu}}g_{\rho'\sigma'} + g^{\rho\rho'}g^{\sigma\sigma'}\frac{\partial g_{\rho'\sigma'}}{\partial g_{\mu\nu}} \\
&= \frac{\partial g^{\rho\rho'}}{\partial g_{\mu\nu}}\delta^{\sigma}_{\rho'} + \frac{\partial g^{\sigma\sigma'}}{\partial g_{\mu\nu}}\delta^{\rho}_{\sigma'} + g^{\rho\rho'}g^{\sigma\sigma'}\delta^{\mu}_{\rho'}\delta^{\nu}_{\sigma'} \\
&= \frac{\partial g^{\rho\sigma}}{\partial g_{\mu\nu}} + \frac{\partial g^{\sigma\rho}}{\partial g_{\mu\nu}} + g^{\rho\mu}g^{\sigma\nu} \\
\implies & \boxed{\frac{\partial g^{\rho\sigma}}{\partial g_{\mu\nu}} = -g^{\rho\mu}g^{\sigma\nu}}
\end{aligned} \tag{107}$$

A.2 Jacobi Formula

For a generic derivative operator ∂, one can show the following two facts:

$$\log\det A = \operatorname{tr}\log A \tag{108}$$

$$\partial \operatorname{tr} F(A) = \operatorname{tr}\left(\frac{d}{dA}F(A)\partial A\right) \tag{109}$$

Then one can prove:

$$\begin{aligned}
\frac{1}{\det A}\partial\det A &= \partial\log\det A = \partial\operatorname{tr}\log A \\
&= \operatorname{tr}\left(\frac{d}{dA}\log A\partial A\right) \\
&= \operatorname{tr}\left(A^{-1}\partial A\right) \\
\implies & \boxed{\partial\det A = \det A\operatorname{tr}\left(A^{-1}\partial A\right)} \\
&= -\det A\operatorname{tr}\left(A(-A^{-2})\partial A\right) \\
& \boxed{\partial\det A = -\det A\operatorname{tr}\left(A\partial(A^{-1})\right)}
\end{aligned} \tag{110}$$

A.3 Metric Density

Let $g \equiv \det[g_{\mu\nu}]$ in this context. We use our result from (110). For variational derivatives w.r.t. to the inverse metric,

$$\begin{aligned} &\boxed{\delta g = -g g_{\mu\nu}\delta g^{\mu\nu}} \\ &\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}} \times (-\delta g) \\ &\boxed{\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}} \end{aligned} \tag{111}$$

For partial derivatives,

$$\begin{aligned} &\boxed{\partial_i g = g g^{ab}\partial_i g_{ba}} \\ &\partial_i\sqrt{-g} = \frac{1}{2\sqrt{-g}}\partial_i(-g) \\ &\boxed{\partial_i\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{ab}\partial_i g_{ab}} \end{aligned} \tag{112}$$

A.4 Connection Coefficients

$$\Gamma^i_{ij} = \frac{1}{2}g^{ia}(\cancel{\partial_i g_{aj}} + \partial_j g_{ia} - \cancel{\partial_a g_{ij}}) \tag{113}$$

$$= \frac{1}{2}g^{ia}\partial_j g_{ia} \tag{114}$$

$$\boxed{\Gamma^i_{ij} = \frac{1}{2g}\partial_j g} \tag{115}$$

$$\text{or } \boxed{\Gamma^i_{ij} = \frac{1}{\sqrt{-g}}\partial_j\sqrt{-g}} \tag{116}$$

$$g^{jk}\Gamma^i_{jk} = \frac{1}{2}g^{jk}g^{ia}(\partial_j g_{ak} + \partial_k g_{ja} - \partial_a g_{jk}) \tag{117}$$

$$= g^{jk}g^{ia}\partial_j g_{ka} - \frac{1}{2}g^{jk}g^{ia}\partial_a g_{jk} \tag{118}$$

$$= \cancel{g^{jk}\partial_j(g^{ia}g_{ka})} - g^{jk}\partial_j g^{ia}g_{ka} - \frac{1}{2}g^{ia}g^{jk}\partial_a g_{jk} \tag{119}$$

$$= -\frac{1}{\sqrt{-g}}\sqrt{-g}\partial_a g^{ia} - \frac{1}{\sqrt{-g}}g^{ia}\partial_a\sqrt{-g} \tag{120}$$

$$\boxed{g^{jk}\Gamma^i_{jk} = -\frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}g^{ia}\right)} \tag{121}$$

A.5 Covariant derivatives

$$\nabla_i v^i = \partial_i v^i + \Gamma^i_{ij} v^j \tag{122}$$

$$= \frac{1}{\sqrt{-g}} \sqrt{-g} \partial_i v^i + \frac{1}{\sqrt{-g}} \partial_j \sqrt{-g} v^j \tag{123}$$

$$\boxed{\nabla_i v^i = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} v^i)} \tag{124}$$

$$\Longrightarrow \boxed{\nabla_i \nabla^i \phi = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} \partial^i \phi)} \tag{125}$$

As a consistency check, we do the divergence of a covector field:

$$g^{ij} \nabla_i \omega_j = g^{ij} \partial_i \omega_j - g^{ij} \Gamma^k_{ij} \omega_k \tag{126}$$

$$= \frac{1}{\sqrt{-g}} \sqrt{-g} g^{ij} \partial_i \omega_j - \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ik}) \omega_k \tag{127}$$

$$\boxed{\nabla^i \omega_i = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \omega_j)} \tag{128}$$

which agrees with (124)

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