Structure of black holes in theories beyond general relativity
LIGO Progress Report 2

Weiming Zhao
Mentors: Leo C. Stein, Vijay Varma, Maria Okounkova

July 6, 2016

Contents

1 Progress

Some of the results I derived in the past month are reproduced below. Refer to the first progress report for the background technical details.

1.1 Inner Product Space of Perturbations

A natural first attempt at an inner product of p_{ab}, q_{cd} in the space of first order stationary, axisymmmetric perturbations of a background metric $g_{ab}^{(0)}$ is

$$\langle p, q\rangle \equiv \int p^{ab} q_{ab} \sqrt{g_{(0)}} d^4x \tag{1.1}$$

$$\langle p, q\rangle = \int dt\, d\phi \int p_{ab}\, g_{(0)}^{ac}\, g_{(0)}^{bd}\, q_{cd} \sqrt{g_{(0)}} d^2x \tag{1.2}$$

where raising and lowering is done by the background metric. Note that in equation (1.2) is only true for stationary, axisymmetric, metrics. The t and ϕ integrals are always the same for all p_{ab} and q_{cd}, so we can factor it out of all inner products.

1.1.1 Trace-reverse and the Inner Product

As a reminder, $\bar{\bar{p}}_{ab} = p_{ab}$, because $\left(p_{ab} - \frac{2}{d} g^{(0)}_{ab} p\right) - \frac{2}{d} g^{(0)}_{ab} \left(p - \frac{2}{d} g^{(0)}_{ab} g^{ab}_{(0)} p\right) = p_{ab}$ and that

$$\bar{p}^{ab}\bar{q}_{ab} = \left(p^{ab} - \frac{2}{d} g^{ab}_{(0)} p\right)\left(q_{ab} - \frac{2}{d} g^{(0)}_{ab} q\right) \tag{1.3}$$

$$= p^{ab}q_{ab} - \frac{2}{d}pq - \frac{2}{d}pq + \frac{4}{d}\cancel{\frac{g^{ab}_{(0)} g^{(0)}_{ab}}{d}} pq \tag{1.4}$$

$$= p^{ab}q_{ab} \tag{1.5}$$

$$\implies \langle p, q\rangle = \langle \bar{p}, \bar{q}\rangle \tag{1.6}$$

1.1.2 Self-Adjointness of the Linearized Einstein Operator

Reading off the form of the linearized Einstein operator $G^{(1)}$ in Lorenz gauge from equation from Progress Report 1,

$$\langle p, G^{(1)}[q]\rangle = \int d^4x \sqrt{g_{(0)}}\, p^{ab} G^{(1)}[q]_{ab} \tag{1.7}$$

$$= \int d^4x \sqrt{g_{(0)}}\, p^{ab}\left(2R^{c\ \ d\ (0)}_{\ a\ b} + \delta^c_a \delta^d_b \Box^{(0)}\right)\bar{q}_{cd} \tag{1.8}$$

$$= \int d^4x \sqrt{g_{(0)}}\left(2R^{a\ \ b\ (0)}_{\ c\ d} p_{ab}\bar{q}^{cd} + p^{cd}\Box^{(0)}\bar{q}_{cd}\right) \tag{1.9}$$

$$= \int d^4x \sqrt{g_{(0)}}\left(\overline{2R^{a\ \ b\ (0)}_{\ c\ d} p_{ab}} q^{cd} + \bar{p}^{cd}\Box^{(0)} q_{cd}\right) \tag{1.10}$$

$$= \int d^4x \sqrt{g_{(0)}}\left(2R^{a\ \ b\ (0)}_{\ c\ d} \bar{p}_{ab} q^{cd} + \bar{p}^{cd}\Box^{(0)} q_{cd}\right) \tag{1.11}$$

where the last step is because we have a Ricci-flat background, so $R^{a\ \ b\ (0)}_{\ c\ d} g^{(0)}_{ab} = 0 = R^{a\ \ b\ (0)}_{\ c\ d} g^{cd}_{(0)}$. And in general, we see that the trace-reverse operator commutes with $G(1)$, i.e. for all q, $\overline{G^{(1)}[\bar{q}]} = G^{(1)}[q]$.

Examining the second term of the integral, we integrate by parts twice and make use of the use the identity from Progress Report 1,

$$\int d^4x \sqrt{g_{(0)}}\, \bar{p}^{cd}\tilde{\nabla}_a\tilde{\nabla}^a q_{cd} = \int d^4x \sqrt{g_{(0)}}\, \tilde{\nabla}_a(\bar{p}^{cd}\tilde{\nabla}^a q_{cd}) - \int d^4x \sqrt{g_{(0)}}\, \tilde{\nabla}_a \bar{p}^{cd}\tilde{\nabla}^a q_{cd} \tag{1.12}$$

$$= \cancel{\int d^4x\, \partial_a(\sqrt{g_{(0)}}\bar{p}^{cd}\tilde{\nabla}^a q_{cd})} - \int d^4x \sqrt{g_{(0)}}\, \tilde{\nabla}_a \bar{p}^{cd}\tilde{\nabla}^a q_{cd} \tag{1.13}$$

$$= -\int d^4x \sqrt{g_{(0)}}\, \tilde{\nabla}^a(\tilde{\nabla}_a \bar{p}^{cd} q_{cd}) + \int d^4x \sqrt{g_{(0)}}\, \tilde{\nabla}^a\tilde{\nabla}_a \bar{p}^{cd} q_{cd} \tag{1.14}$$

$$= -\cancel{\int d^4x\, \partial_a(\sqrt{g_{(0)}}\tilde{\nabla}^a \bar{p}^{cd} q_{cd})} + \int d^4x \sqrt{g_{(0)}}\, \tilde{\nabla}^a\tilde{\nabla}_a \bar{p}^{cd} q_{cd} \tag{1.15}$$

Therefore, we have

$$\langle p, G^{(1)}[q]\rangle = \int d^4x \sqrt{g_{(0)}}\left(2R^{a\ \ b\ (0)}_{\ c\ d} + \delta^a_c \delta^b_d \Box^{(0)}\right)\bar{p}_{ab} q^{cd} \tag{1.16}$$

$$= \int d^4x \sqrt{g_{(0)}}\, G^{(1)}[p]^{cd} q_{cd} \tag{1.17}$$

$$= \langle G^{(1)}[p], q\rangle \tag{1.18}$$

The operator $G^{(1)}$ is self-adjoint with respect to this inner product.

1.2 Weyl-Lewis-Papapetrou (WLP)

In order to prove this we need a little machinery called Frobenius' Theorem.

1.2.1 Frobenius' Theorem

There are a few equivalent statements of Frobenius' Theorem; while the differential form version is nice, we use the vector field form for our current purposes. Frobenius' Theorem is useful not only for the proof of uniqueness of the WLP metric, but also will be used to show the integrability conditions for the solution to the Einstein Field Equations under a WLP metric.

Without introducing to many definitions, the theorem is roughly

Theorem 1.1 *In order to have a smooth sub-manifold of $\mathcal{M}$ that has tangent spaces coinciding with a tangent sub-bundle $W \subseteq E$ over $\mathcal{M}$, it is necessary and sufficient for W to be involute, i.e. $\forall X^a, Y^a \in W : [X,Y]^a \in W$.*

Therefore we have the following corollary:

Corollary 1.1.1 *If vector fields X^a and Y^a commute, with either vanishing at a point, and $X^a R_a{}^{[b} X^c Y^{d]} = 0 = Y^a R_a{}^{[b} Y^c X^{d]}$, then the 2-fold orthogonal to X^a and Y^a are integrable.*

The proofs are outlined in Wald[6], and may be reproduced here at a later time.

1.2.2 Proof of WLP

Given a time-like $\left(\frac{\partial}{\partial t}\right)^a$ and an "azimuthal" space-like $\left(\frac{\partial}{\partial \phi}\right)^a$ Killing vector fields for stationary axisymmetric $1+3$ dimensional spacetimes. These satisfy corollary 1.1.1, so the span of the other vector fields generated by the other two coordinates (x_2 and x_3) are orthogonal to ∂_t^a and ∂_ϕ^a.

$$ds^2 = V(x_2,x_3)dt^2 + 2W(x_2,x_3)dt d\phi + X(x_2,x_3)d\phi^2 + g_{ij}(x_2,x_3)dx^i dx^j \tag{1.19}$$

for $i,j \in \{2,3\}$. In block matrix form, the metric is

$$g_{ab} = \begin{pmatrix} -V & W & 0 & 0 \\ W & X & 0 & 0 \\ 0 & 0 & g_{22} & g_{23} \\ 0 & 0 & g_{23} & g_{33} \end{pmatrix} \tag{1.20}$$

Note that there are six distinct functions of x_2 and x_3.

We choose $x_2 = \rho = VX + W^2$, which is the negative of determinant of the upper 2×2 block. And choose $x_3 = z$ be such that $\nabla_a \rho \nabla^a z = 0$. Redefining variables, we must have

$$ds^2 = -V(dt - w d\phi)^2 + V^{-1}\rho^2 d\phi^2 + \Omega^2(d\rho^2 + \Lambda dz^2) \tag{1.21}$$

where $w = W/V$, $\Omega^2 = g_{22}$, and $\Lambda = g_{33}/\Omega^2$.

The four functional degrees of freedom are $V(\rho,z), w(\rho,z), \Omega(\rho,z), \Lambda(\rho,z)$.

We have made a gauge transformation to the unique Weyl-Lewis-Papapetrou coordinates for any stationary, axisymmetric spacetime, up to univariate scaling of z.

1.3 Schwarzschild in Weyl-Lewis-Papapetrou

1.3.1 Schwarzschild Background

We want to describe spacetimes in with a Schwarzschild background. Therefore we expect there to exist $V = V_0 + \delta V, w = w_0 + \delta w, \Omega = \Omega_0 + \delta\Omega, \Lambda = \Lambda_0 + \delta\Lambda$, where the variables with the naught-subscripts describe Schwarzschild background metric, and the δ variables are perturbations that keep the metric stationary and axisymmetric. Let's solve for the Schwarzschild solution only in terms of the background first, with no perturbations; we need to get the metric into the form:

$$ds^2 = -V_0(dt - w_0 d\phi)^2 + V_0^{-1}\rho^2 d\phi^2 + \Omega_0^2(d\rho^2 + \Lambda_0 dz^2) \tag{1.22}$$

Note that at the end of our calculation, we expect to choose coordinates so that $\Lambda_0 = 1$ because Schwarzschild is Ricci-flat.

1.3.2 Motivation of WLP Coordinates

By Birkhoff's Theorem, the Schwarzschild metric is axisymmetric and stationary (in fact it is static):

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.23}$$

Therefore we should be able to write the metric in Weyl-Lewis-Papapetrou form.

We keep the time and azimuthal directions the same, as it is natural to pick $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ as our Killing vector fields. Therefore were are transforming the spatial coordinates r and θ only, from those that are spherically symmetric to those cylindrically symmetric.

We identify that $V_0 = 1 - \frac{2m}{r}$ and $w_0 = 0$, so our metric is in the form:

$$ds^2 = -V_0(dt - w_0 d\phi)^2 + V_0^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.24}$$

We see that the standard spherical to cylindrical ($r\sin\theta \mapsto \rho$, $r\cos\theta \mapsto z$) will not suffice because the only $d\phi^2$ term in the line element will be $r^2\sin^2\theta d\phi^2 \mapsto \rho^2 d\phi^2$, and in the WLP form, we need $V_0^{-1}\rho^2 d\phi^2$. Thus, we make our transformation $V_0^{1/2} r\sin\theta \mapsto \rho$, so that $r^2\sin^2\theta d\phi^2 \mapsto V_0^{-1}\rho^2 d\phi^2$.

Our transformation is so far defined by

$$\rho = V_0^{1/2} r\sin\theta = \sqrt{r^2 - 2mr}\sin\theta \tag{1.25}$$

$$\implies d\rho = \frac{r - m}{V_0^{1/2} r}\sin\theta dr + \underbrace{V_0^{1/2} r\cos\theta}_{\tilde{\rho}}\, d\theta \tag{1.26}$$

We see that $\tilde{\rho} = V_0^{1/2} r\cos\theta$ is the trigonometric conjugate of $\rho = V_0^{1/2} r\sin\theta$ (i.e. $\tilde{\rho}^2 + \rho^2 = V_0 r^2$). And with a clever definition of z, we have

$$z = (r - m)\cos\theta \tag{1.27}$$

$$\implies dz = \cos\theta dr - \underbrace{(r - m)\sin\theta}_{\tilde{z}}\, d\theta \tag{1.28}$$

where $\tilde{z} = (r - m)\sin\theta$ is the trignometric conjugate of $z = (r - m)\cos\theta$.

We see a good sign that $\frac{\tilde{z}}{V_0^{1/2} r}$ appears in (1.26) and $\frac{\tilde{\rho}}{V_0^{1/2} r}$ appears in (1.28).

So with this transformation:

$$\boxed{\begin{aligned} &t = t && (1.29)\\ &\rho = V_0^{1/2} r\sin\theta = \sqrt{r^2 - 2mr}\sin\theta && (1.30)\\ &z = (r - m)\cos\theta && (1.31)\\ &\phi = \phi && (1.32)\end{aligned}}$$

we have

$$\boxed{\begin{aligned} &dt = dt \\ &d\rho = V_0^{-1/2} r^{-1} \tilde{z} dr + \tilde{\rho} d\mathrm{z} \\ &dz = V_0^{-1/2} r^{-1} \tilde{\rho} dr - \tilde{z} d\theta \\ &d\phi = d\phi \end{aligned}} \qquad \begin{aligned} &(1.33) \\ &(1.34) \\ &(1.35) \\ &(1.36) \end{aligned}$$

Therefore, we have in terms of the auxiliary variables $\tilde{\rho} = V_0^{1/2} r \cos\theta$ and $\tilde{z} = (r-m)\sin\theta$,

$$\implies \tilde{z} d\rho + \tilde{\rho} dz = V_0^{-1/2} r^{-1} (\tilde{z}^2 + \tilde{\rho}^2) dr \tag{1.37}$$

$$\implies \boxed{dr = \frac{V_0^{1/2} r}{\tilde{z}^2 + \tilde{\rho}^2} (\tilde{z} d\rho + \tilde{\rho} dz)} \tag{1.38}$$

$$\implies \tilde{\rho} d\rho - \tilde{z} dz = (\tilde{z}^2 + \tilde{\rho}^2) d\theta \tag{1.39}$$

$$\implies \boxed{d\theta = \frac{1}{\tilde{z}^2 + \tilde{\rho}^2} (\tilde{\rho} d\rho - \tilde{z} dz)} \tag{1.40}$$

Substituting into the metric,

$$ds^2 = -V_0 (dt - w_0 d\phi)^2 + V_0^{-1} \rho^2 d\phi^2 + \cancel{V_0^{-1}} \frac{\cancel{V_0} r^2}{(\tilde{z}^2 + \tilde{\rho}^2)^2} (\tilde{z} d\rho + \tilde{\rho} dz)^2 + r^2 \frac{(\tilde{\rho} d\rho - \tilde{z} dz)^2}{(\tilde{z}^2 + \tilde{\rho}^2)^2} \tag{1.41}$$

$$ds^2 = -V_0 (dt - w_0 d\phi)^2 + V_0^{-1} \rho^2 d\phi^2 + \frac{r^2}{(\tilde{z}^2 + \tilde{\rho}^2)^{\cancel{2}}} \left(\cancel{(\tilde{z}^2 + \tilde{\rho}^2)} d\rho^2 + \cancel{(\tilde{z}^2 + \tilde{\rho}^2)} dz^2 \right) \tag{1.42}$$

$$ds^2 = -V_0 (dt - w_0 d\phi)^2 + V_0^{-1} \rho^2 d\phi^2 + \frac{r^2}{\tilde{z}^2 + \tilde{\rho}^2} \left(d\rho^2 + dz^2 \right) \tag{1.43}$$

We see that we've chosen z correctly so that $\Lambda_0 = 1$ and

$$\Omega_0^2 = \frac{r^2}{\tilde{z}^2 + \tilde{\rho}^2} = \frac{r^2}{(r^2 - 2mr + m^2)\sin^2\theta + (r^2 - 2mr)\cos^2\theta} \tag{1.44}$$

$$= \frac{r^2}{(r^2 - 2mr) + m^2 \sin^2\theta} \tag{1.45}$$

Therefore we have for the Schwarzschild background

$$ds^2 = -V_0 (dt - w_0 d\phi)^2 + V_0^{-1} \rho^2 d\phi^2 + \Omega_0^2 \left(d\rho^2 + \Lambda_0 dz^2 \right) \tag{1.46}$$

So our Weyl-Lewis-Papapetrou functional degrees of freedom are, as functions (r, θ),

$$\boxed{\begin{aligned} &V = \left(1 - \frac{2m}{r}\right) + \delta V \\ &w = 0 + \delta w \\ &\Omega^2 = \frac{r^2}{(r^2 - 2mr) + m^2 \sin^2\theta} + \delta\Omega^2 \\ &\Lambda = 1 + \delta\Lambda \end{aligned}} \qquad \begin{aligned} &(1.47) \\ &(1.48) \\ &(1.49) \\ &(1.50) \end{aligned}$$

1.3.3 Coordinate Singularities of Background Schwarzschild

Despite the curvature singularity at $r = 0$, we have coordinate singularities when $\Omega_0^2 \to \infty$, i.e.

$$0 = r^2 - 2mr + m^2 \sin^2\theta \tag{1.51}$$
$$0 = (r-m)^2 - m^2\cos^2\theta \tag{1.52}$$
$$0 = (\underbrace{r - m + m\cos\theta}_{R_+})(\underbrace{r - m - m\cos\theta}_{R_-}) \tag{1.53}$$

With the auxiliary variables $R_\pm$, we rewrite our WLP functions with the substitution $r = \frac{1}{2}(R_+ + R_- + 2m)$:

$$\boxed{\begin{aligned} V &= \frac{R_+ + R_- - 2m}{R_+ + R_- + 2m} + \delta V && (1.54)\\ w &= 0 + \delta w && (1.55)\\ \Omega^2 &= \frac{(R_+ + R_- + 2m)^2}{4R_+R_-} + \delta\Omega^2 && (1.56)\\ \Lambda &= 1 + \delta\Lambda && (1.57)\end{aligned}}$$

$$\begin{aligned}
\rho^2 + z^2 &= (r^2 - 2mr)\sin^2\theta + (r-m)^2\cos^2\theta && (1.58)\\
&= (r-m)^2 + m^2\cos^2\theta - m^2 && (1.59)\\
&= (r - m \pm m\cos\theta)^2 - m^2 \mp 2(r-m)m\cos\theta && (1.60)\\
&= R_\pm^2 - m^2 \mp 2mz && (1.61)\\
\implies \rho^2 + (z \pm m)^2 &= R_\pm^2 && (1.62)\\
\implies & \boxed{R_\pm = \sqrt{\rho^2 + (z\pm m)^2}} && (1.63)
\end{aligned}$$

and thus our WLP functions are now functions of (ρ, z).

The coordinate singularities corresponding to $R_\pm = 0$ are now at $(\rho, z) = (0, \pm m)$ for all t and ϕ.

We also have a coordinate singularity when $\rho \to 0$, so all the coordinate singularities are at the line $\rho = 0$ in the spacetime, which includes the $(\rho, z) = (0, \pm m)$ singularity as well.

1.4 Mathematica for perturbations of Kerr and Schwarzschild

I was able to calculate the Einstein operator in WLP coordinates for both a Kerr and Schwarzschild backgrounds. The Kerr solution in WLP form I used are from Jones and Wang[3]. The solutions with the explicit coordinates are too long to reproduce here in the progress report.

1.4.1 Lorenz Gauge

I have shown in Mathematica perturbations off of Schwarzschild in WLP do not correspond to Lorenz gauge, therefore, WLP and Lorenz gauge are not the same gauge. We speculate that WLP is a generalized harmonic gauge though.

1.5 Bianchi Identity

1.5.1 General Connections

Baez and Muniain[1] outline an elegant proof of the Bianchi identity, reproduced here in detail. We will use the the Bianchi identity to show the geometric origin of the divergencelessness of the Einstein tensor and all possible source terms.

Given a fiber bundle $\pi : E \to \mathcal{M}$ and a connection D on $\mathcal{M}$, for any E-valued form $\eta = s_I \otimes \omega^I$ on $\mathcal{M}$, in local coordinates,

$$\begin{aligned}
\mathrm{d}_D^2\eta &= \mathrm{d}_D\left(D_\nu s_I \otimes \mathrm{d}x^\nu \wedge \mathrm{d}x^I\right) && (1.64)\\
&= D_\mu D_\nu s_I \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (1.65)\\
&= \frac{1}{2}\left[D_\mu, D_\nu\right] s_I \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (1.66)\\
&= \frac{1}{2} F_{\mu\nu} s_I \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (1.67)\\
&= F \wedge \eta && (1.68)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{d}_D^3\eta &= \mathrm{d}_D(\mathrm{d}_D^2\eta) && (1.69)\\
&= \mathrm{d}_D(F \wedge \eta) && (1.70)\\
&= \mathrm{d}_D F \wedge \eta + F \wedge \mathrm{d}_D\eta && (1.71)\\
\mathrm{d}_D^3\eta &= \mathrm{d}_D^2(\mathrm{d}_D\eta) && (1.72)\\
&= F \wedge \mathrm{d}_D\eta && (1.73)\\
\implies & \boxed{\mathrm{d}_D F = 0} && (1.74)
\end{aligned}$$

In local coordinates,

$$\begin{aligned}
0 = \mathrm{d}_D F \wedge \eta &= \mathrm{d}_D\left(\frac{1}{2} F_{\mu\nu} \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right) \wedge \left(s_I \otimes \mathrm{d}x^I\right) && (1.75)\\
&= \frac{1}{2}(D_\lambda F_{\mu\nu}) \otimes \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \left(s_I \otimes \wedge \mathrm{d}x^I\right) && (1.76)\\
&= \frac{1}{2}(D_\lambda F_{\mu\nu}) s_I \otimes \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (1.77)\\
&= \frac{1}{2}\left(D_\lambda\left(F_{\mu\nu} s_I\right) - F_{\mu\nu}\left(D_\lambda s_I\right)\right) \otimes \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (1.78)\\
&= \frac{1}{2}[D_\lambda, F_{\mu\nu}] s_I \otimes \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (1.79)\\
&= \frac{1}{2} \cdot \frac{1}{3}\left([D_\lambda, F_{\mu\nu}] + [D_\mu, F_{\nu\lambda}] + [D_\nu, F_{\lambda\mu}]\right) s_I \otimes \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^I && (1.80)\\
\implies 0 &= [D_\lambda, F_{\mu\nu}] + [D_\mu, F_{\nu\lambda}] + [D_\nu, F_{\lambda\mu}] && (1.81)\\
&\boxed{0 = [D_\lambda, [D_\mu, D_\nu]] + [D_\mu, [D_\nu, D_\lambda]] + [D_\nu, [D_\lambda, D_\mu]]} && (1.82)
\end{aligned}$$

which is in the form of the Jacobi identity.

1.5.2 With Riemann curvature

For our Levi-Civita connection ∇ compatible with metric g, we have the curvature $R(u,v)w = \left([\nabla_u, \nabla_v] - \nabla_{[u,v]}\right) w$, which is just the curvature of the connection ∇.

$$\begin{aligned}
0 &= [u,[v,w]] + [v,[w,u]] + [w,[u,v]] && (1.83)\\
&= \nabla_u[v,w] - \nabla_{[v,w]}u + (uvw \text{ cyc}) && (1.84)\\
&= \nabla_u(\nabla_v w - \nabla_w v) - \nabla_{[v,w]}u + (uvw \text{ cyc}) && (1.85)\\
&= [\nabla_u, \nabla_v]w - \nabla_{[u,v]}w + (uvw \text{ cyc}) && (1.86)\\
0 &= R(\nabla_u, \nabla_v)w + (uvw \text{ cyc}) && (1.87)\\
& && (1.88)
\end{aligned}$$

Specifically, we adopt the convention for the Riemann curvature tensor, $R_{abc}{}^{d}e_d \equiv R(\nabla_a, \nabla_b)e_c$. Choose $u = \partial_a, v = \partial_b, w = \partial_c$ to be coordinate basis vector fields.

$$\implies 0 = R(\nabla_a, \nabla_b)\partial_c + R(\nabla_b, \nabla_c)\partial_a + R(\nabla_c, \nabla_a)\partial_b \tag{1.89}$$

$$\implies 0 = R_{abc}{}^{d} + (abc \text{ cyc}) \tag{1.90}$$

$$0 = R^{d}{}_{cab} + (abc \text{ cyc}) \tag{1.91}$$

$$0 = R^{d}{}_{abc} + (abc \text{ cyc}) \tag{1.92}$$

$$\implies \boxed{0 = R^{d}{}_{[abc]}} \tag{1.93}$$

From eq. (1.82) applied to the Levi-Civita connection,

$$0 = [\nabla_a, [\nabla_b, \nabla_c]] + [\nabla_b, [\nabla_c, \nabla_a]] + [\nabla_c, [\nabla_a, \nabla_b] \tag{1.94}$$

$$= [\nabla_a, R(\nabla_b, \nabla_c)]e_d + (abc \text{ cyc}) \tag{1.95}$$

$$= \nabla_a R^{e}{}_{bcd} e_e - \cancel{R(\nabla_b, \nabla_c) g_{ad}} + (abc \text{ cyc}) \tag{1.96}$$

$$\implies 0 = \nabla_a R_{ebcd} + (abc \text{ cyc}) \tag{1.97}$$

$$\implies 0 = \nabla_a R_{ecdb} + \nabla_a R_{edbc} + (abc \text{ cyc}) \tag{1.98}$$

where we use eq. (1.92) in the last step.

Contracting with the metric twice,

$$0 = g^{ec} \left(\cancel{\nabla_a R_{ecdb}} + \nabla_a R_{edbc} + (abc \text{ cyc})\right) \tag{1.99}$$

$$0 = -\nabla_a R_{db} + \nabla_b R_{da} + \nabla^e R_{edab} \tag{1.100}$$

$$0 = g^{bd} \left(-\nabla_a R_{db} + \nabla_b R_{da} + \nabla^e R_{edab}\right) \tag{1.101}$$

$$0 = -\nabla_a R + \nabla^d R_{da} + \nabla^e R_{ea} \tag{1.102}$$

$$\implies 0 = \nabla^d \underbrace{(2R_{da} - g_{da} R)}_{2G_{da}} \tag{1.103}$$

$$\implies \boxed{0 = \nabla^d G_{da}} \tag{1.104}$$

1.6 The action of Einstein operator in WLP gauge: Ricci-flat

This part was quite difficult, even for with the Ricci-flat simplification. The manipulations here are not referenced anywhere and it took a lot of sweat and trial and error to get the following result.

1.6.1 Constraint equations

From $G_{ab} = 0$, we have ostensibly 6 non-zero equations of motion, which correspond to G_{00}, G_{03}, G_{33} and G_{11}, G_{12}, G_{22}.

From the first three, we can construct the combinations

$$e^{2\gamma} \left(\left(V^{-2} - \rho^{-2} w^2\right) G_{00} + e^{2\gamma} \rho^{-2} w^2\right) G_{33} = \vec{\nabla} \left(V^{-1} \vec{\nabla} V + \rho^{-2} V^2 w \vec{\nabla} w\right) \tag{1.105}$$

$$e^{2\gamma} \rho^{-2} \left(w G_{00} + G_{03}\right) = \vec{\nabla} \left(\rho^{-2} V^2 \vec{\nabla} w\right) \tag{1.106}$$

where $\vec{\nabla}$ is the gradient under the flat metric $ds^2 = \rho^2 d\phi^2 + d\rho^2 + dz^2$, *not* $ds^2 = g_{ab} dx^a dx^b$

We have $G_{00} = G_{03} = G_{33} = 0$ if and only if

$$0 = \vec{\nabla} \left(V^{-1} \vec{\nabla} V + \rho^{-2} V^2 w \vec{\nabla} w\right) \tag{1.107}$$

$$0 = \vec{\nabla} \left(\rho^{-2} V^2 \vec{\nabla} w\right) \tag{1.108}$$

and the Bianchi identity $\nabla^a G_{ab} = 0$ is satisfied.

Furthermore, we have $G_{11} = -G_{22}$ automatically, so we are left with

$$0 = -G_{11} = G_{22} = \frac{1}{4V^2}\left((\partial_\rho V)^2 - (\partial_z V)^2\right) - \frac{V^2}{4\rho^2}\left((\partial_\rho w)^2 - (\partial_z w)^2\right) - \frac{\partial_\rho \gamma}{\rho^2} \tag{1.109}$$

$$0 = G_{12} = \frac{\partial_z \gamma}{\rho^2} - \frac{1}{2V^2}(\partial_\rho V)(\partial_z V) + \frac{V^2}{2\rho^2}(\partial_\rho w)(\partial_z w) \tag{1.110}$$

which are compatible because given eqs. (1.107) and (1.108), $\partial_\rho \partial_z \gamma = \partial_z \partial_\rho \gamma$ is true.

We have shown that there are 4 equations (2 of which are compatible) consistent with 3 metric variables in the Ricci-flat case.

1.7 Numerics

I have begun work on the numerics, the next phase of my project.

2 Challenges

There are some goals I have set and challenges I face:

- Complete the WLP analysis for the non-Ricci-flat case
- Express constraints in terms of perturbations and background
- Check if ADM equations are automatically satisfied or need to be constrained during the numerical evolution.
- Invert Linearized equation
- Work on the relaxation code and numerics
- Figuring out correct boundary conditions and compactifying coordinates to bring in infinity.
- Show whether WLP is the best gauge to calculate in or if we need to transform into a spherical gauge
- Show whether WLP is a generalized harmonic gauge.

References

[1] John C. Baez and J.P. Muniain. *Gauge Fields, Knots, and Gravity.* K & E series on knots and everything. World Scientific, 1994.

[2] David Hilditch. An introduction to well-posedness and free-evolution. 2013.

[3] Gregory Jones and John Wang. Weyl card diagrams. 2005.

[4] Andrea Maselli, Paolo Pani, Leonardo Gualtieri, and Valeria Ferrari. Rotating black holes in einstein-dilaton-gauss-bonnet gravity with finite coupling. 2015.

[5] Leo C. Stein. Rapidly rotating black holes in dynamical chern-simons gravity: Decoupling limit solutions and breakdown. 2014.

[6] R.M. Wald. *General Relativity.* University of Chicago Press, 1984.

[7] Kent Yagi and Leo C. Stein. Black hole based tests of general relativity. 2016.