LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
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Simplifying DARM response with a small SRC detuning

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1 Overview

In this document, we show that when the signal recycling cavity is detuned by a small amount, the DARM response as a function of frequency f can be written in the form of

$$\frac{\partial S}{\partial L_-}(f) = \frac{He^{-2i\pi f\tau}}{1 + if/f_{\rm cc}} \times \left(\frac{f^2}{f^2 + f_{\rm s}^2 - iff_{\rm s}/Q} \right), \tag{1}$$

where H, $f_{\rm cc}$, τ, $f_{\rm s}$, and Q are the optical gain, DARM coupled cavity pole, time delay, optical spring frequency, and optical spring Q factor, respectively.

2 Derivation

2.1 Master equation

We start from equation (3.83) of Ward's thesis [1] for the DARM response. Subsequently, we will gradually approximate it to a simpler form. The original equation is given in a fairly complicated form of

$$\frac{\partial S}{\partial L_-}(\omega) = \frac{t_{\rm s}\left[\left(1 - r_{\rm s}e^{2i\beta}\right)\cos\zeta\cos\phi - \left(1 + r_{\rm s}e^{2i\beta}\right)\sin\zeta\sin\phi\right]}{1 + r_{\rm s}^2 e^{4i\beta} - 2r_{\rm s}e^{2i\beta}\left[\cos 2\phi + \frac{\kappa}{2}\sin 2\phi\right]}\sqrt{2I_{\rm bs}\frac{\omega_0^2}{\omega^2\left(\omega_{\rm c}^2 + \omega^2\right)}}, \tag{2}$$

where

$$\kappa = \frac{8I_{\rm bs}}{mL^2}\frac{\omega_0}{\omega^2\left(\omega_{\rm c}^2 + \omega^2\right)}, \tag{3}$$

and $t_{\rm s}$ and $r_{\rm s}$ are the transmissivity and reflectivity of the signal recycling mirror, and where ω, ω_0 and $\omega_{\rm c}$ are the gravitational wave, laser and single-arm cavity pole angular-frequencies, and ζ and ϕ are homodyne and detuning phases, $\beta = -\arctan\frac{\omega}{\omega_{\rm c}}$ is the phase delay in the arm cavity, $I_{\rm bs}$, m and L are the laser power on the beam splitter, the mass of the test masses and the length of the arm cavities. Figure 1 shows the RSE response function with various detuning settings.

Each test mass is assumed to have the same mass, m and assumed to be a free mass.

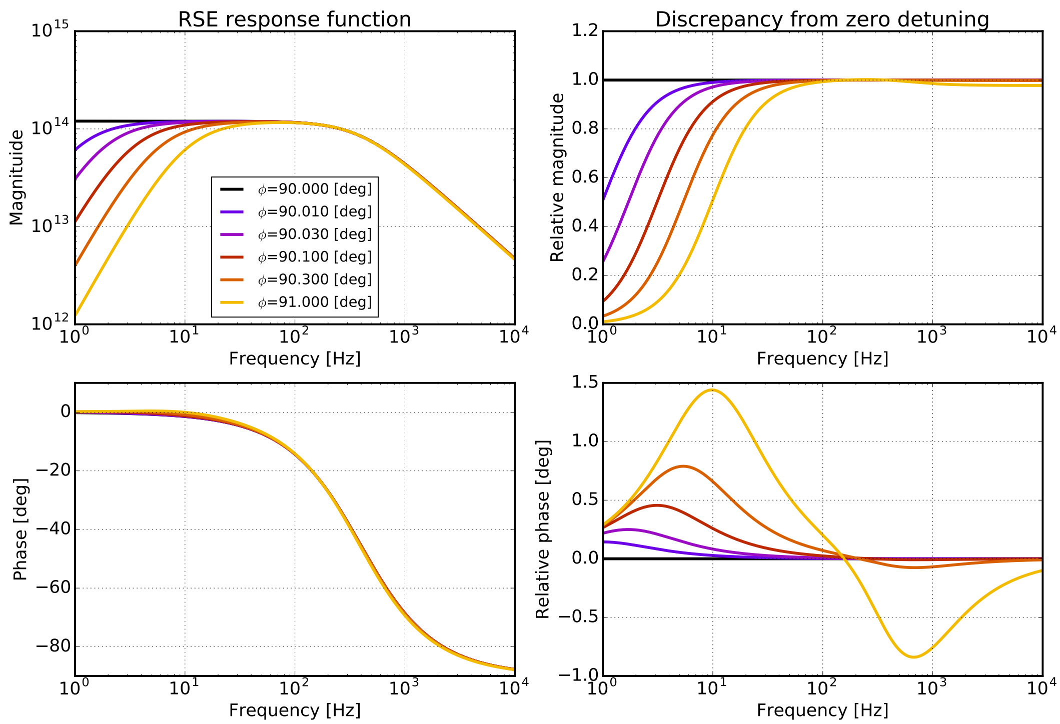

Figure 1: The frequency response of the detuned RSE with various detuning settings. The following numerical parameters are used to compute the response: $\omega_\mathrm{c} = 2\pi \times 45\,\mathrm{rad/s}$; $t_\mathrm{s} = \sqrt{0.37}$; $r_\mathrm{s} = \sqrt{1-0.37}$; $\zeta = \pi/2$; $m = 40\,\mathrm{kg}$; $L = 3994.5\,\mathrm{m}$; $I_\mathrm{bs} = 1600\,\mathrm{W}$.

2.2 Approximation

According to a first study by Hall *et al.* [2], it seems that the following approximations can work.

- 90° homodyne phase ($\zeta \to \pi/2$).
- a small detuning in the signal recycling cavity ($\phi \to \pi/2 + \Delta\phi$).

Applying the first approximation ($\zeta \to \pi/2$) to equation (2), we obtain

$$\frac{\partial S}{\partial L_-} = -\frac{t_\mathrm{s}\left(1 + r_\mathrm{s} e^{2i\beta}\right)\sin\phi}{1 + r_\mathrm{s}^2 e^{4i\beta} - 2 r_\mathrm{s} e^{2i\beta}\left[\cos 2\phi + \frac{\kappa}{2}\sin 2\phi\right]}\sqrt{2 I_\mathrm{bs}\frac{\omega_0^2}{\omega^2\left(\omega_\mathrm{c}^2 + \omega^2\right)}}$$

Now, we apply the second approximation ($\phi \to \pi/2 + \Delta\phi$) to the last equation,

$$\frac{\partial S}{\partial L_-} = -\frac{t_\mathrm{s}\left(1 + r_\mathrm{s} e^{2i\beta}\right)}{1 + r_\mathrm{s}^2 e^{4i\beta} + 2 r_\mathrm{s} e^{2i\beta}\left[1 + \kappa\Delta\phi\right]}\sqrt{2 I_\mathrm{bs}\frac{\omega_0^2}{\omega^2\left(\omega_\mathrm{c}^2 + \omega^2\right)}}.$$

This equation should be equivalent to the usual single-pole response [3] when there is no detuning (i.e., $\Delta\phi \to 0$).

Our strategy here is to remove the single-pole response by dividing it out of the last equation and leave the remaining function, M, which is governed by the optical spring effect. The optical spring effect M can be computed as

$$\begin{aligned} M(\omega) &= \frac{\partial S}{\partial L_-}\left(\left.\frac{\partial S}{\partial L_-}\right|_{\Delta\phi=0}\right)^{-1} \\ &= \frac{1}{1 + A\kappa\Delta\phi}, \end{aligned}$$

where A is a complex number defined as $A = 2r_\mathrm{s}e^{2i\beta} / \left(1 + r_\mathrm{s}^2 e^{4i\beta} + 2r_\mathrm{s}e^{2i\beta}\right)$. Plugging equation (3) to the last equation, one can rewrite it as

$$M(\omega) = \frac{\omega^2}{\omega^2 + \frac{Z}{(\omega_\mathrm{c}^2 + \omega^2)}},$$

where Z is a complex frequency-dependent number defined as $Z = 8 A I_\mathrm{bs}\omega_0\Delta\phi/(mL^2)$. According to the measurements (e.g., [2]) and model (see figure 1), a small detuning slightly rotates the phase at low frequencies. By looking at the magnitude and phase components of Z, and assuming a small phase component of Z, we can simplify the detuning function M:

$$M(\omega) = \frac{\omega^2}{\omega^2 + \frac{|Z|e^{i\theta}}{\omega_c^2 + \omega^2}} \tag{4}$$

$$= \frac{\omega^2}{\omega^2 + \frac{|Z|\cos\theta}{\omega_c^2 + \omega^2} + \frac{i|Z|\sin\theta}{\omega_c^2 + \omega^2}} \tag{5}$$

$$\approx \frac{\omega^2}{\omega^2 + \frac{|Z|}{\omega_c^2 + \omega^2} + \frac{i|Z|\theta}{\omega_c^2 + \omega^2}} \tag{6}$$

Returning to the defintion of Z, the phase of Z must equal the phase of A. A is a function of the SRC reflectivity r_s and arm cavity phase delay β. The arm cavity phase delay β is itself a function of frequency ω:

$$\theta = \arg(Z) = \arg(A) = 2\beta = -2\arctan\frac{\omega}{\omega_c} \tag{7}$$

$$M(\omega) \approx \frac{\omega^2}{\omega^2 + \frac{|Z|}{\omega_c^2+\omega^2} - i\,\frac{2|Z|\arctan(\omega/\omega_c)}{\omega_c^2+\omega^2}} \tag{8}$$

We now assume that as the frequency ω grows, the condition $|Z|/(\omega_c^2+\omega^2) \ll \omega^2$ is satisfied before the condition $\omega_c^2 \ll \omega^2$ is satisfied. This allows the simplifications $(\omega_c^2+\omega^2) \rightarrow \omega_c^2$ and $\arctan\frac{\omega}{\omega_c} \rightarrow \frac{\omega}{\omega_c}$. Therefore, the optical spring effect can be written

$$M(\omega) = \frac{\omega^2}{\omega^2 + \omega_s^2 - i\,\omega\,\omega_s/Q} \tag{9}$$

where ω_s and Q are a real constants defined by

$$\omega_s^2 = \frac{|Z|}{\omega_c^2} \qquad Q = \frac{\omega_c}{2\omega_s} \tag{10}$$

For an anti-spring detuning, $\omega_s^2 > 0$ whereas it should be $\omega_s^2 < 0$ for a pro-spring detuning.

In summary, the DARM response with a small signal recycling detuning can be written as

$$S(f) = \frac{He^{-2i\pi f\tau}}{1 + if/f_{cc}} \times M(f). \tag{11}$$

which is what we have showed at the beginning in equation (1).

3 Accuracy

3.1 Full Model vs. Simplified Model

Running numerical simulations, we found that the accuracy of the approximated function [equation (1)] was good to sub-percent levels in magnitude and sub-degree levels in phase for a detuning of 0.5 deg.

Figure 2 shows a comparison of the approximated optical spring [equation (9)] and that of the full form [equation (2)]. An excellent agreement is shown in the figure at an accuracy

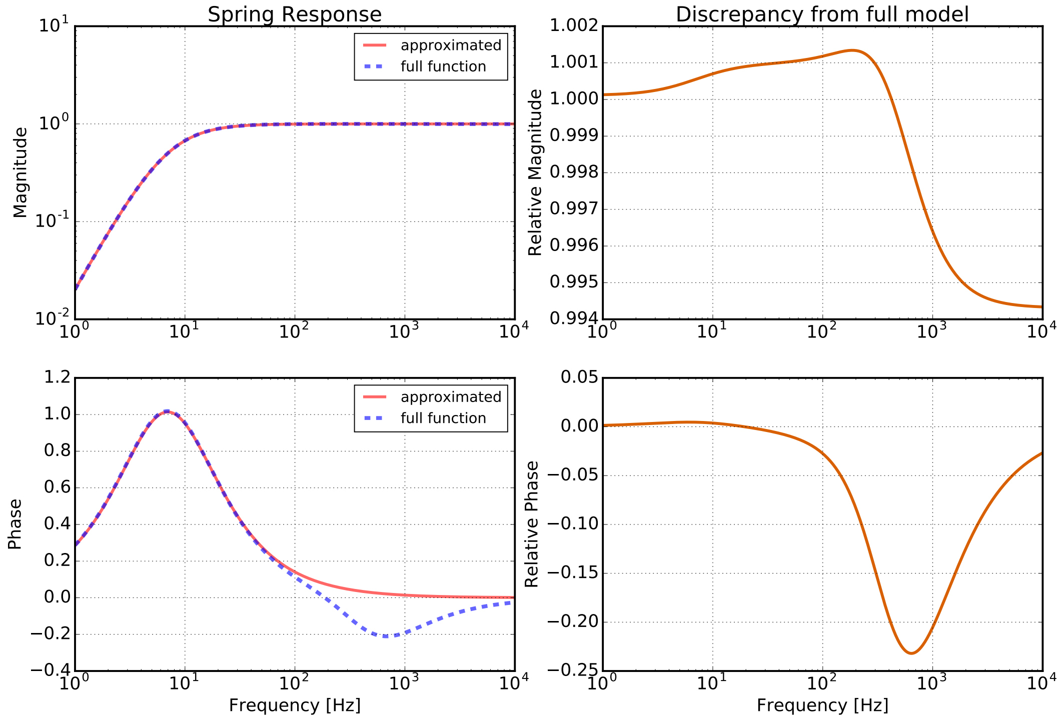

Figure 2: The frequency response of the detuned RSE with various detuning settings. $\phi = 90.5\,\text{deg}$ (or $\Delta\phi = 0.5\,\text{deg}$) is assumed. The other numerical parameters are the same as that used in figure 1. A best fit for f_s was found to be 6.96 Hz, while a best fit for Q was found at 28.25

level of sub percent in magnitude. Discrepancy in the magnitude above 300 Hz seems to be due to the cavity pole slightly moved by the detuning.

At higher frequencies our assumptions based on $\omega_c^2 \ll \omega^2$ break down, leading to slight discrepancies from the full model.

3.2 ER9 Hanford Detuning

The Hanford interferometer plant has been measured twice for Engineering Run 9. Just as in O1, Hanford is demonstrating detuning at low frequencies. What's more, it appears the detuning changes in the days between the two measurements. This is a strong argument for including a calibration line at low frequencies in the Hanford detector to monitor detuning.

Using the `emcee` python package, we have fit a model using Equation 1 to both ER9

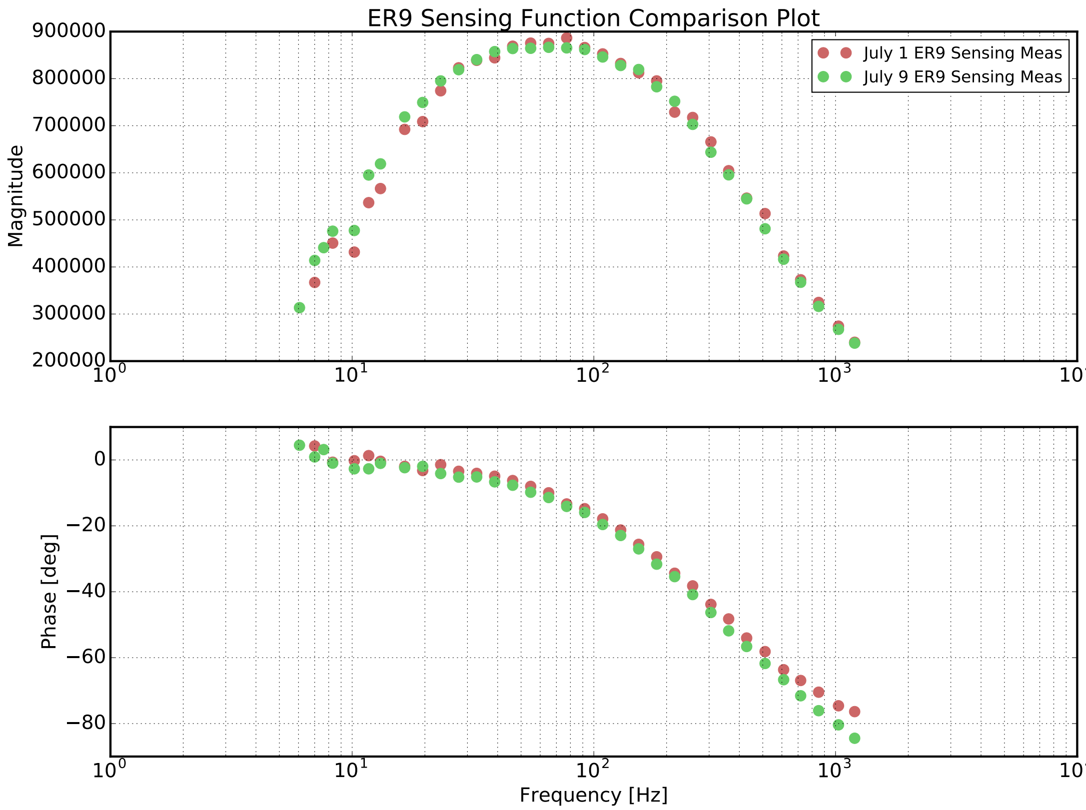

Figure 3: The ER9 LHO sensing function measurements from July 1st and July 9th.

measurements separately. The results of the MCMC fits can be seen in Figures 4 and 5. Note that we fit a parameter called inverse Q for simplicity; inverse Q is allowed to be 0 without blowing up the denominator of equation 1.

The ER9 detuned sensing model changes the parameters significantly. The detuning changes from lock to lock, as captured by the value of the optical spring frequency f_s. The July 1st $f_s = 9.97$ Hz, but the July 9th $f_s = 8.30$ Hz. These optical spring frequencies correspond to a detuning phase of $\Delta\phi = 1.03$ degrees and $\Delta\phi = 0.71$ degrees respectively.

The other parameters see alterations as well. The ER9 LHO coupled cavity pole is 320 Hz whereas during O1 the nominal LHO coupled cavity pole was 341 Hz. Time delay sees a 15 μsec increase between the July 1st and July 9th ER9 measurements. The optical spring inverse Q reduces by a factor of 2.7 between the July 1st and July 9th.

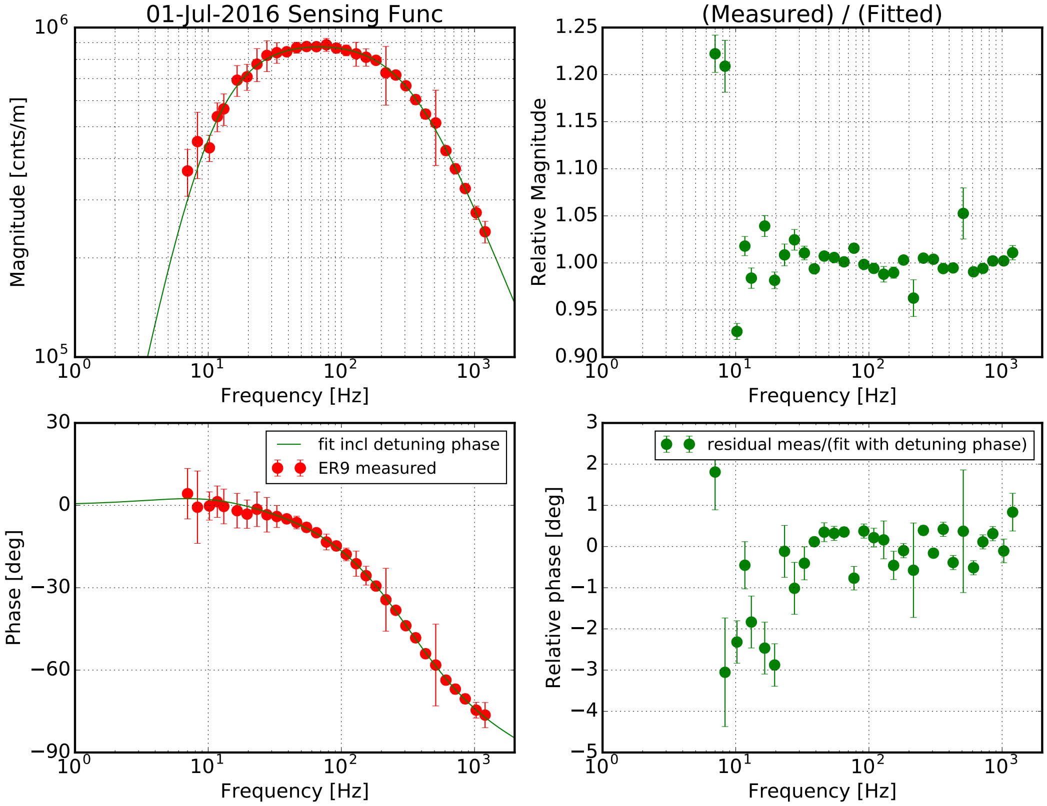

Figure 4: MCMC Fit of the July 1st ER9 LHO sensing function measurement.
Parameter Values:

```
Optical gain = 9.12e+05 +/- 8.16e+02 [cnts/m]
Cavity pole = 3.23e+02 +/- 5.51e-01 [Hz]
Time delay = 5.46e+00 +/- 3.46e-01 [μsec]
Spring frequency = 9.97e+00 +/- 5.49e-02 [Hz]
Spring Inverse Q = 1.36e-01 +/- 3.52e-03
```

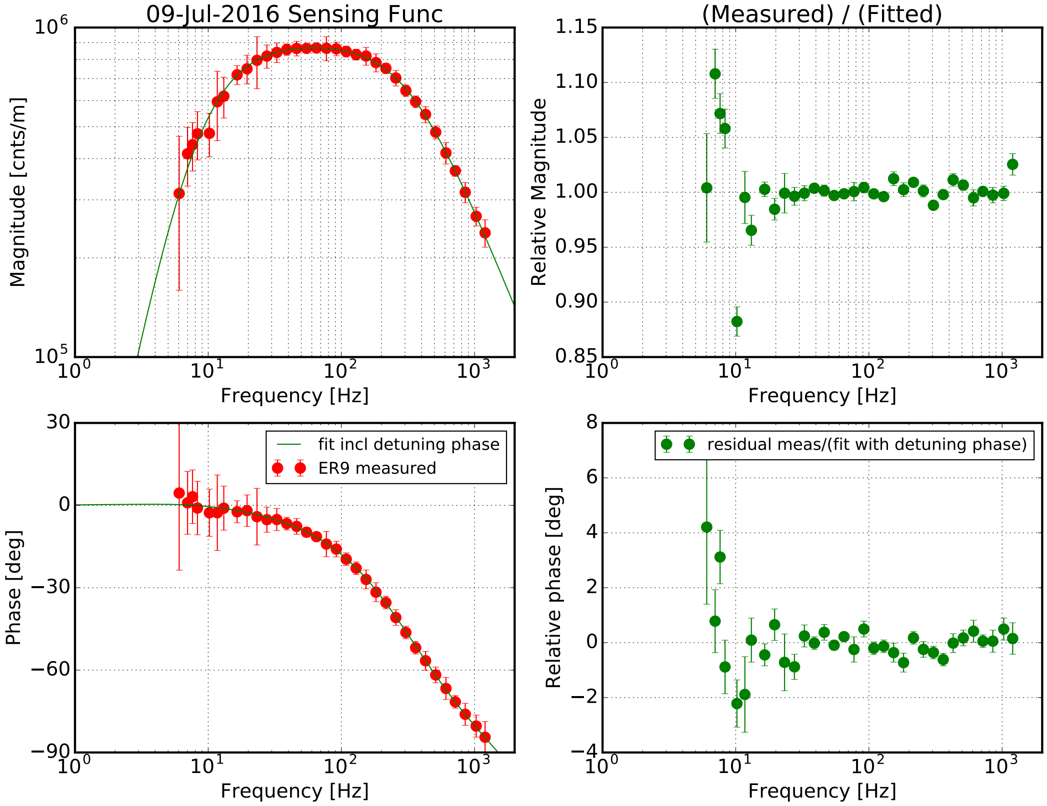

Figure 5: MCMC Fit of the July 9th ER9 LHO sensing function measurement.
Parameter Values:

```
Optical gain = 8.99e+05 +/- 1.09e+03 [cnts/m]
Cavity pole = 3.20e+02 +/- 8.47e-01 [Hz]
Time delay = 2.21e+01 +/- 5.33e-01 [μsec]
Spring frequency = 8.30e+00 +/- 6.15e-02 [Hz]
Spring Inverse Q = 5.10e-02 +/- 6.79e-03
```

References

[1] Robert Ward, "Length Sensing and Control of an Advanced Prototype Interferometric Gravitational Wave Detector", Thesis, California Institute of Technology (2010
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[2] Evan Hall, "0.5° of SRC detuning can explain O1 anomaly in DARM plant", LHO alog 27675 (2016)
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[3] Kiwamu Izumi "Calibration meeting material: toward accurate DARM response modeling", LIGO-G1501316-v1
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