Measuring Kerrness in Binary Black Hole Simulation Ringdowns
Progress Report 2
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After a single black hole forms from the merger of a black hole binary, it enters the ringdown phase, radiating away energy in gravitational waves until it settles into a stationary Kerr black hole. However, the point at which the resultant spacetime becomes close to Kerr has not yet been established. Furthermore, how close to the merger phase in a gravitational waveform can LIGO apply data analysis techniques that assume that the remnant black hole is Kerr (or a perturbation thereof)? In order to address these questions, it is helpful to consider local quantities which measure the similarity of a spacetime to Kerr. We evaluate several local measures of "Kerrness" on volume data from the ringdown phase of a binary black hole simulation of the GW150914 event using the Spectral Einstein Code (SpEC). We also evaluate these quantities on single (Kerr) black hole simulations for validation.

$$S = \frac{27J^2}{I^3} \tag{1}$$

FIG. 1: Definition for the speciality index [2]. This quantity is 1 for a Kerr spacetime.

$$\mathcal{L} \equiv \frac{(\mathfrak{r}(A)+\mathfrak{r}(B))^2 + (\mathfrak{j}(A)_i+\mathfrak{j}(B)_i)(\mathfrak{j}(A)^i+\mathfrak{j}(B)^i) + (\mathfrak{t}(A)_{ij}+\mathfrak{t}(B)_{ij})(\mathfrak{t}(A)^{ij}+\mathfrak{t}(B)^{ij})}{\sigma^{14}} + \frac{\mathfrak{a}_{ij}\mathfrak{a}^{ij}+\mathfrak{b}_{ij}\mathfrak{b}^{ij}}{\sigma^4} + \frac{((1-3\lambda^2)\beta+\lambda(3-\lambda^2)\alpha)^2}{\sigma^2} + \frac{(\mathfrak{B}_{ij}\mathfrak{B}^{ij})^3}{\sigma^4} + \frac{(\mathfrak{C}_{ij}\mathfrak{C}^{ij})^3}{\sigma^7} + \frac{\Omega}{\sigma^2} \quad (2)$$

FIG. 2: Positive Kerrness invariant proposed by Gómez-Lobo [3]. This quantity vanishes for Kerr initial data, and is being investigated as a measurement of the Kerrness of a spacetime. Compute items for the first three terms have been created and computed on single black hole simulations. The first, fourth, and last terms are coded but display deviations from expected behavior and convergence properties, which are believed to be caused by subtle problems with the code or dependencies. The fifth term (not highlighted) is not yet implemented in a compute item, but all of its dependencies have been implemented.

PROGRESS EVALUATION

Code organization, which was a goal for the previous month, was improved to some extent, but was not as high a priority as fixing run time problems with certain compute items. Better organizational practices were applied when creating new and editing previously created compute items. After all compute items are confirmed to behave well with regard to convergence, further organization of the code will occur as time permits

The speciality index (S) was computed on binary black hole ringdown data from a simulation of GW150914 as a function of radius and decomposed into spherical harmonics. A plot of the real part of the Y_0^0 mode of $Re(S)$ is shown in FIG. 3.

Quantities newly implemented include $\Theta^{\parallel}$, $R^{\parallel}$, $\Theta^{\perp}_{\mu}$, $R^{\perp}_{\mu}$, M, Y, Y_j, $\mathfrak{r}(\Xi)$, $\mathfrak{j}(\Xi)$, $\mathfrak{t}(\Xi)$, $\mathfrak{B}_{ij}$, and Ω (see appendix).

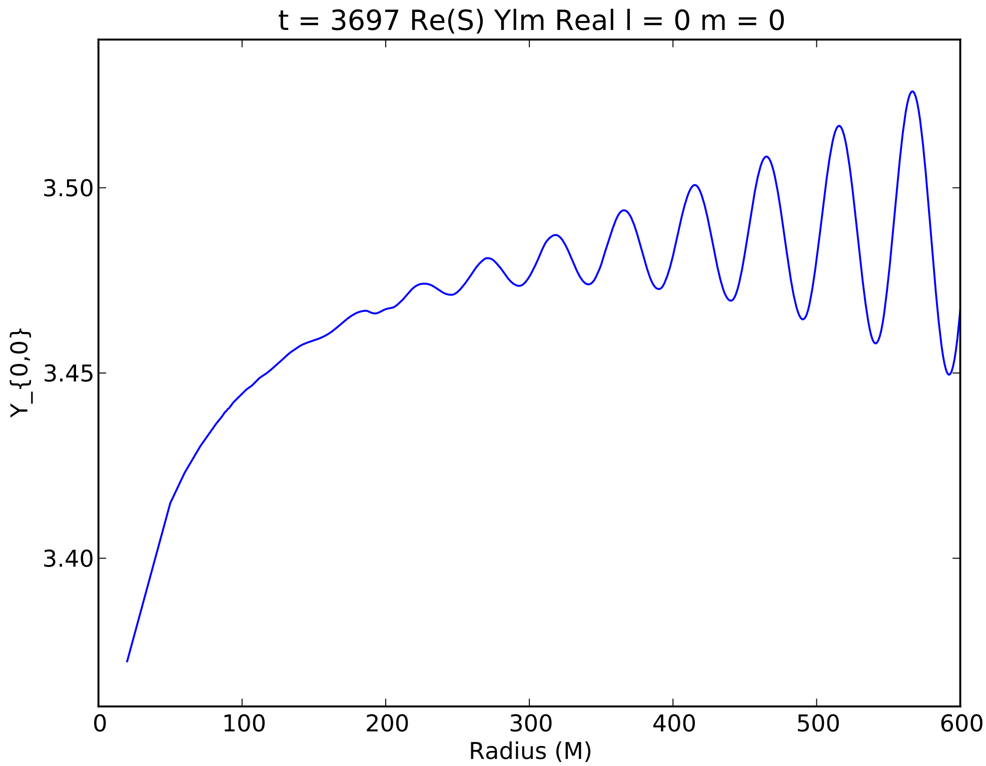


FIG. 3: The real part of the (0, 0) spherical harmonic mode of the real part of S as a function of radius at the first time step (3697M) of the ringdown in the GW150914 binary black hole simulation.

CHALLENGES

Thus far

A consistency check between evaluation of the expression for the intermediate quantity $\Theta^{\|}$[3] between equation (3) (equation (66) [3])

$$\Theta^{\|} = \frac{1}{A^2+B^2}\Big((2B(E^{\beta\alpha}E^{\gamma}{}_{\alpha} - B^{\beta\alpha}B^{\gamma}{}_{\alpha}) + 4AB^{\beta\alpha}E^{\gamma}{}_{\alpha})K_{\beta\gamma} - \frac{2}{3}\varepsilon_{\beta\gamma\delta}(E^{\alpha\beta}(BD^{\delta}B_{\alpha}{}^{\gamma} + AD^{\delta}E_{\alpha}{}^{\gamma}) + B^{\alpha\beta}(-AD^{\delta}B_{\alpha}{}^{\gamma} + BD^{\delta}E_{\alpha}{}^{\gamma}))\Big) \tag{3}$$

and equation (4) (equation (71) [3]) in the proof

$$\Theta^{\parallel} \equiv \frac{1}{3}(A^2+B^2)^{-1}(A\pounds_{\vec{n}}B - B\pounds_{\vec{n}}A) \tag{4}$$

yielded different results as summarized in FIG. 4 (a). The cause of this inconsistency was found to be a sign error in one of the terms in equation (66) as described below:

From equation (71)[3],

$$\Theta^{\parallel} = \frac{1}{3}\left(A^2+B^2\right)^{-1}\left(A\pounds_{\vec{n}}B - B\pounds_{\vec{n}}A\right) \tag{5}$$

where $\pounds_{\vec{n}}A$, $\pounds_{\vec{n}}B$ are given in (57) and (58)[3]

$$\pounds_{\vec{n}}A = 6(B_\alpha{}^\gamma B^{\alpha\beta} - E_\alpha{}^\gamma E^{\alpha\beta})K_{\beta\gamma} + 4AK^\gamma{}_\gamma + 2\epsilon_{\beta\gamma\delta}(E^{\alpha\beta}D^\delta B_\alpha{}^\gamma + B^{\alpha\beta}D^\delta E_\alpha{}^\gamma) \tag{6}$$

$$\pounds_{\vec{n}}B = -4(3B^{\alpha\beta}E_\alpha{}^\gamma K_{\beta\gamma} - BK^\gamma{}_\gamma) + 2\epsilon_{\beta\gamma\delta}(B^{\alpha\beta}D^\delta B_\alpha{}^\gamma - E^{\alpha\beta}D^\delta E_\alpha{}^\gamma) \tag{7}$$

Simplifying the result in equation (8) (sign difference from eq. (66)[3] highlighted)

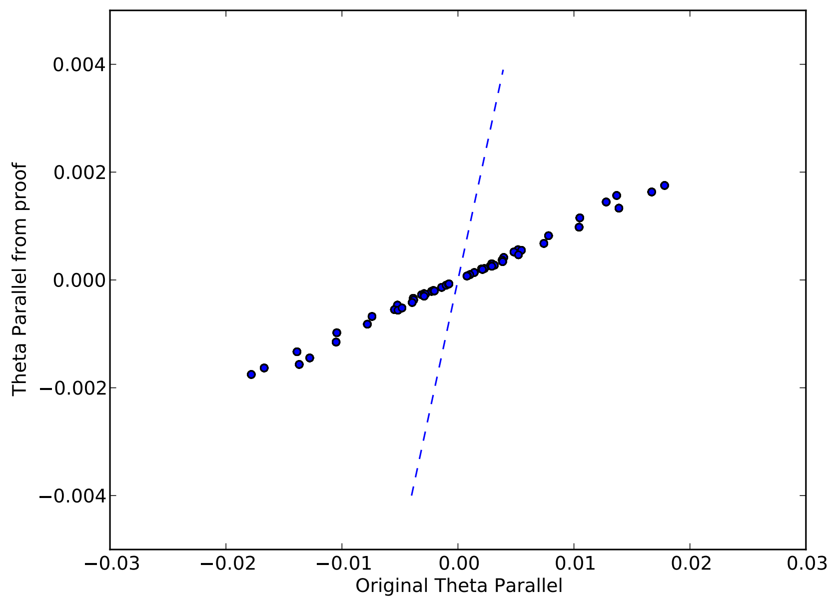
$$\begin{aligned}\Theta^{\parallel} &= \frac{1}{3}\left(A^2+B^2\right)^{-1}\big(-12AB^{\alpha\beta}E_\alpha{}^\gamma K_{\beta\gamma} + 4ABK^\gamma{}_\gamma + 2A\epsilon_{\beta\gamma\delta}(B^{\alpha\beta}D^\delta B_\alpha{}^\gamma - E^{\alpha\beta}D^\delta E_\alpha{}^\gamma) \\ &\qquad - 6B(B_\alpha{}^\gamma B^{\alpha\beta} - E_\alpha{}^\gamma E^{\alpha\beta})K_{\beta\gamma} - 4ABK^\gamma{}_\gamma - 2B\epsilon_{\beta\gamma\delta}(E^{\alpha\beta}D^\delta B_\alpha{}^\gamma + B^{\alpha\beta}D^\delta E_\alpha{}^\gamma)\big) \\ &= \left(A^2+B^2\right)^{-1}\Bigg((2B(E_\alpha{}^\gamma E^{\alpha\beta} - B_\alpha{}^\gamma B^{\alpha\beta}) - 4AB^{\alpha\beta}E_\alpha{}^\gamma)K_{\beta\gamma} \\ &\qquad - \frac{2}{3}\epsilon_{\beta\gamma\delta}(E^{\alpha\beta}(BD^\delta B_\alpha{}^\gamma + AD^\delta E_\alpha{}^\gamma) + B^{\alpha\beta}(BD^\delta E_\alpha{}^\gamma - AD^\delta B_\alpha{}^\gamma))\Bigg)\end{aligned} \tag{8}$$

Note that due to the symmetries of the Electric and Magnetic Weyl tensors E_{ij} and B_{ij}, $B^{\alpha\beta}E_\alpha{}^\gamma$ is equivalent to $B^{\beta\alpha}E^\gamma{}_\alpha$, and similarly for other seemingly index-swapped terms.

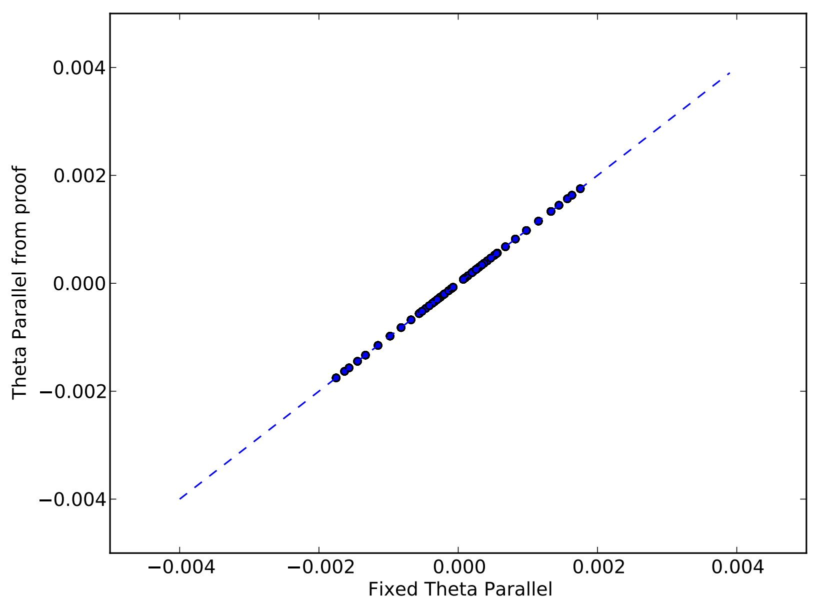
In addition, several minor errors in the implementation which are not the result of mathematical errors have been found and fixed. Compute items fixed include `GomezLoboThetaPerp` ($\Theta^\perp_\mu$), `GomezLoboRPerp` ($R^\perp_\mu$), `GomezLoboD` (D), and `GomezLoboK` (K).

Anticipated and Ongoing

Convergence problems appear to be most pronounced in quantities involving second derivatives.



(a)



(b)

FIG. 4: Scatter plots of the original (a) and fixed (b) $\Theta^{\|}$ quantities against the quantity from the proof for Sphere 1 (6M radius). Because the values for $\Theta^{\|}$ are analytically identical to one another, the scatter plot points should lie on the dashed line along y = x.

From (84)[3],

$$M \equiv \left(\frac{\alpha}{1 - 3\lambda^2} \right)^{\frac{2}{3}} \tag{9}$$

Currently, in some regions of the simulation, the quantity $\frac{\alpha}{1-3\lambda^2}$ is negative, leading to floating point errors when evaluating the quantity of the fractional exponent 2/3.

When computing the terms of $\mathcal{L}$ on the GW150914 binary black hole ringdown volume data, changes to the input files will be required to correctly account for the frame in which to compute derivatives. It is possible to neglect details such as these when computing quantities on the single black hole simulation, and so there has not yet been an opportunity to test for these errors.

GOALS

Now that most of the $\mathcal{L}$ quantity has been implemented, it may be run on the binary black hole ringdown volume data from the GW150914 simulation. The better behaved terms in the expression may be run first while the rest are investigated further.

One specific goal to achieve greater code organization is to combine the compute items related to $\mathcal{L}$ into one source file.

The $\mathcal{L}$ quantity is mostly coded aside from one remaining term and some bug fixes. The terms which have been coded and behave well on the single black hole simulation will be run on the binary black hole ringdown volume as the rest of the terms are implemented or fixed. Once all terms are coded, $\mathcal{L}$ in its entirety can be run on the ringdown data.

In addition, a simplified and analogous quantity proposed by Gómez-Lobo in a more recent paper [4] has been implemented in SpEC by Maria Okounkova [5]. A goal is to compare the behavior of this quantity with $\mathcal{L}$ in single black hole simulations and on the GW150914 ringdown volume data.

At this time, it seems unlikely that time will permit the creation, debugging and verification of compute items for the Kerrness measure proposed by Bäckdahl and Valiente Kroon [1], and thus goals for the remainder of the project have shifted towards more thoroughly analyzing simulations involving compute items which already exist.

APPENDIX

The following quantities [3] have been implemented as compute items in the past month:

$$M \equiv \left(\frac{\alpha}{1-3\lambda^2}\right)^{\frac{2}{3}} , \tag{10}$$

$$Y^2 = \frac{\mathfrak{r}(\Xi)}{M} , \tag{11}$$

$$Y_j = \frac{-\mathfrak{j}(\Xi)_j}{MY} , \tag{12}$$

$$\mathfrak{B}_{ij} \equiv D_{(i}Y_{j)} - YK_{ij} , \tag{13}$$

$$\Omega \equiv (\mathfrak{t}(\Xi)_{ij} - MY_iY_j)(\mathfrak{t}(\Xi)^{ij} - MY^iY^j) = 0 , \tag{14}$$

$$\Theta^{\|} = \frac{1}{A^2+B^2}\Bigg((2B(E^{\beta\alpha}E^{\gamma}{}_{\alpha} - B^{\beta\alpha}B^{\gamma}{}_{\alpha}) + 4AB^{\beta\alpha}E^{\gamma}{}_{\alpha})K_{\beta\gamma} - \frac{2}{3}\varepsilon_{\beta\gamma\delta}(E^{\alpha\beta}(BD^{\delta}B_{\alpha}{}^{\gamma} + AD^{\delta}E_{\alpha}{}^{\gamma}) + B^{\alpha\beta}(-AD^{\delta}B_{\alpha}{}^{\gamma} + BD^{\delta}E_{\alpha}{}^{\gamma}))\Bigg) , \tag{15}$$

$$R^{\|} = -\frac{4}{3}K^{\gamma}{}_{\gamma} + \frac{1}{A^2+B^2}\Bigg((2A(E^{\beta\alpha}E^{\gamma}{}_{\alpha} - B^{\beta\alpha}B^{\gamma}{}_{\alpha}) + 4BB^{\beta\alpha}E^{\gamma}{}_{\alpha})K_{\beta\gamma} - \frac{2}{3}\varepsilon_{\beta\gamma\delta}(B^{\alpha\beta}(BD^{\delta}B_{\alpha}{}^{\gamma} + AD^{\delta}E_{\alpha}{}^{\gamma}) + E^{\alpha\beta}(AD^{\delta}B_{\alpha}{}^{\gamma} - BD^{\delta}E_{\alpha}{}^{\gamma}))\Bigg) , \tag{16}$$

$$R^{\perp}{}_{\mu} \equiv \frac{1}{6}D_{\mu}(\log(A^2+B^2)) , \tag{17}$$

$$\Theta^{\perp}{}_{\mu} \equiv \frac{1}{3}\left(A^2+B^2\right)^{-1}(BD_{\mu}A - AD_{\mu}B) \tag{18}$$

$$\mathfrak{r}(\Xi) \equiv \frac{1}{(\lambda^2-1)\sqrt{A^2+B^2}} \times (X(\Pi)_{\mu\nu}(-\Theta^{\perp\mu}\Theta^{\perp\nu} + R^{\perp\mu}R^{\perp\nu}) - 2\mathcal{E}(Q)_{\mu\nu}R^{\perp\mu}\Theta^{\perp\nu})) , \tag{19}$$

$$\mathfrak{j}(\Xi)_{\mu} \equiv \frac{-1}{(\lambda^2-1)\sqrt{A^2+B^2}} \times \Bigg(\Theta^{\perp\nu}(\mathcal{E}(Q)_{\mu\nu}R^{\|} + X(\Pi)_{\mu\nu}\Theta^{\|} - \varepsilon_{\mu\rho\alpha}Z(\Pi)_{\nu}{}^{\alpha}\Theta^{\perp\rho}) - R^{\perp\nu}(R^{\|}X(\Pi)_{\mu\nu} - \mathcal{E}(Q)_{\mu\nu}\Theta^{\|} + \varepsilon_{\mu\nu\alpha}\ \mathcal{B}(Q)_{\rho}{}^{\alpha}\Theta^{\perp\rho} + \varepsilon_{\mu\rho\alpha}(-R^{\perp\rho}Z(\Pi)_{\nu}{}^{\alpha} + \mathcal{B}(Q)_{\nu}{}^{\alpha}\Theta^{\perp\rho}))\Bigg) , \tag{20}$$

$$
\mathfrak{t}(\Xi)_{\mu\nu} \equiv \frac{1}{(\lambda^2-1)\sqrt{A^2+B^2}} \times
$$

$$
\Big(X(\Pi)^{\rho}{}_{\rho}(R^{\perp}_{\mu}R^{\perp}_{\nu} - \Theta^{\perp}_{\mu}\Theta^{\perp}_{\nu}) - 2\mathcal{E}(Q)_{\mu\nu}(R^{\parallel}\Theta^{\parallel} + R^{\perp\rho}\Theta^{\perp}_{\rho}) -
$$

$$
2\varepsilon_{\rho p(\nu}(\mathcal{B}(Q)_{\mu)}{}^{\rho}(R^{\perp p}\Theta^{\parallel} + R^{\parallel}\Theta^{\perp p}) + \mathcal{E}(Q)_{\mu)}{}^{\rho}(-R^{\parallel}R^{\perp p} + \Theta^{\parallel}\Theta^{\perp p})) +
$$

$$
2\mathcal{E}(Q)_{\rho(\nu}(R^{\perp\rho}\Theta^{\perp}_{\mu)} + R^{\perp}_{\mu)}\Theta^{\perp\rho}) + 2X(\Pi)_{\rho(\nu}(-R^{\perp}_{\mu)}R^{\perp\rho} + \Theta^{\perp}_{\mu)}\Theta^{\perp\rho}) +
$$

$$
X(\Pi)_{\mu\nu}(R^{\parallel 2} + R^{\perp}_{\rho}R^{\perp\rho} - \Theta^{\parallel 2} - \Theta^{\perp}_{\rho}\Theta^{\perp\rho}) +
$$

$$
h_{\mu\nu}(X(\Pi)^{\alpha}{}_{\alpha}(-R^{\perp}_{\rho}R^{\perp\rho} + \Theta^{\perp}_{\rho}\Theta^{\perp\rho}) - 2\mathcal{E}(Q)_{\rho\alpha}R^{\perp\rho}\Theta^{\perp\alpha} + X(\Pi)_{\rho\alpha}(R^{\perp\rho}R^{\perp\alpha} - \Theta^{\perp\rho}\Theta^{\perp\alpha}))\Big)\,, \tag{21}
$$

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[2] J. Baker and M. Campanelli. Making use of geometrical invariants in black hole collisions. *Physical Review D*, 62(12):127501, December 2000.

[3] A. García-Parrado Gómez-Lobo. Local non-negative initial data scalar characterization of the Kerr solution. *Physical Review D*, 92(12):124053, December 2015.

[4] A. García-Parrado Gómez-Lobo. Vacuum type d initial data. *ArXiv e-prints*, February 2016.

[5] Maria Okounkova. Private Communication, 2016.