Detecting and Measuring Eccentric Black Hole Binaries

Interim Report 1, LIGO Student Undergraduate Research Fellowship, California Institute of Technology

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Abstract

Due to emission of gravitational wave radiation, most compact binaries are expected to circularize before emitting GW in the LIGO frequency band (GW frequency from 20 to 3000 Hz, corresponding to orbital frequency of 10 to 1500 Hz). However, if a binary black hole system resulted from dynamical capture or hierarchal triple interactions close to the end of its life, there is a probability that the system could still be in eccentric orbit while in the LIGO band. As such, observing eccentricity from a gravitational wave signal could be a clear signature of dynamical origins. We seek to discover if the LIGO detectors are capable of detecting gravitational radiation from BBH in eccentric orbit. We will assess the deviation of eccentric waveforms from quasi-circular waveforms in the time and frequency domains, i.e. how much SNR will be lost when using a quasi-circular waveform template to detect a BBH with eccentricity. Using Bayesian inference, we will first work in a one-dimensional parameter space to extract the eccentricity parameter from a GW signal, assuming all other parameters are known. Then, we will work in a three dimensional parameter space to identify chirp mass, mass ratio, and eccentricity, and search for degeneracies amongst these parameters, as well as with spin. The results from this project will indicate how much eccentricity affects measurements of GW strain from BBH. Additionally, results will help enable important insight into the nature of formation mechanisms of binary black holes and the environments in which they reside: do they more typically evolve in isolated binaries, as indicated by circularized orbits, or are they more often a result from dynamical interactions in dense stellar environments?

I. INTRODUCTION

To date, LIGO's gravitational wave detectors have observed gravitational waves from binary black holes that follow a consistent pattern: the BBH systems have quasi-circular orbits that decrease in radius and increase in frequency as they lose energy in the form of gravitational radiation. All observed gravitational waves from BBH fit a "chirp" waveform while within the LIGO frequency band; their amplitude and frequency increase as the binary evolves [11]. The plots in Figure 1 show predicted chirp waveforms; Figure 2 shows an "actual" waveform, or bandpassed, filtered data from GW150914, LIGO's first detected gravitational wave. Although General Relativity predicts subtle effects from eccentricity, LIGO data analysis methodology presently uses waveform templates that assume a negligible eccentricity; highly eccentric BBH could go undetected with current technology. Additionally, many properties of BBH are currently calculated with the assumption of circular orbits, such as their distance from Earth [11]. Identifying eccentricity could make such calculations more accurate, and will yield a better understanding BBH formation mechanisms and the stellar environments in which they reside.

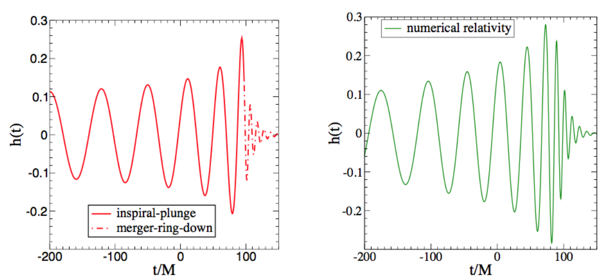

FIG. 1. Chirp waveform predicted using effective-one-body approach (left) and numerical relativity simulations (right) for same initial conditions. Figure from [11]

Binary black holes have several formation mechanisms, the two most widely understood being isolated binary evolution and dynamical interaction in dense stellar environments such as globular clusters or galactic nuclei. Evolutionary trajectories for both formation mechanisms predict that under most circumstances, the orbits of BBH systems will have circularized by the time their emitted gravitational radiation is within the LIGO band. However, if a BBH forms via dynamical capture with an extremely large eccentricity and/or extremely close to the end of its lifetime (i.e., a small periapse), there *is* a probability that

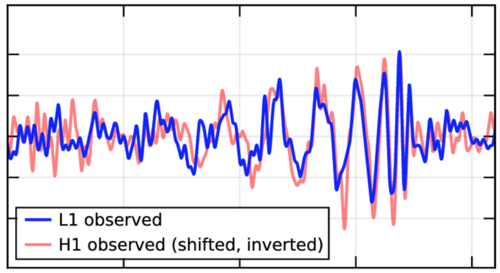

FIG. 2. Strain (h(t)) plotted against time (t) for GW150914, LIGO's first GW detection. This plot shows data that has been filtered, i.e. subject to band-passing and line removal. L1 is the data from the Livingston, Louisiana detector; H1 is the data from the Hanford, Washington detector. Figure from [6]

the system could be detected using quasi-circular templates while still in a non-circular orbit [1]. Another possibility for observing an eccentric BBH in the LIGO band is via a hierarchal triple, a quasi-stable three-body system where one BBH is orbited by another black hole. Ellipticity can be produced in hierarchal triples through angular momentum exchange from the inner binary and the larger system, in what is known as the Kozai-Lidov mechanism[9], [7].

Dense stellar regions, such as galactic nuclei and globular clusters that have undergone core collapse, are prime spots for dynamical BH-BH capture. In such settings, individual black holes can become gravitationally bound during close passage as energy is lost in the form of a GW burst[4]. Samsing predicts that up to 5% of the observable BBH forming in globular clusters have an eccentricity > 0.1 [10]. Once a dynamically captured BH pair is in eccentric orbit, a GW burst is theorized to be emitted every time the pair passes at a close encounter (i.e. at its periapse). This causes the periapse (r_p) and eccentricity (e) to decrease with time, while orbital frequency increases with time. After sufficient energy is lost through gravitational radiation, the BH pair will merge.

Several groups have created waveform models for eccentric BBH. For example, East et al. created a model combining inspiral and merger models to predict a full waveform for dynamical capture binaries. Their results are shown in Figure 3. This model can be applied where standard Post-Newtownian waveforms fail, i.e., the late stages of merger [4]. This waveform model has not been implemented in LIGO's template bank.

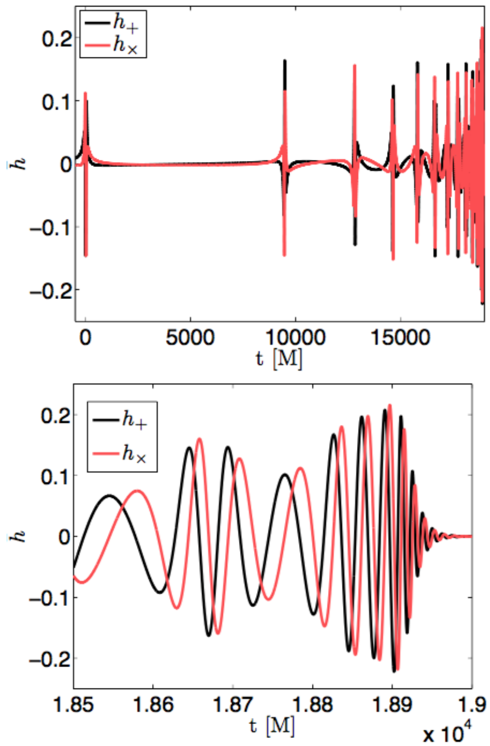

FIG. 3. Gravitational wave strain plot generated by the model described in East et al. for a periapse $r_p = 0.8M$ and an eccentricity e = 1. The top panel shows the entire waveform; the bottom panel shows a zoomed-in view of the end of the waveform. Figure from [4]

II. OBJECTIVES

The aim of this project is to assess detectability and identifiability of eccentric binary black holes. We will focus on three groups of questions:

1. What are key features of eccentric waveforms in the time and frequency domains? In Python, we will create a waveform family for a range of initial eccentricities using a lowest order Parametrized Post-Newtonian approximation and compare these waveforms to existing models in LALSuite. As a function of phase difference, how do these eccentric waveforms deviate from quasi-circular waveforms?

2. Does eccentricity make GW signals from BBH harder to detect? Specifically, how much SNR can be lost if a search pipeline uses quasi-circular binary templates to capture an eccentric binary merger event? We will calculate this as a function of the initial eccentricity (i.e. eccentricity at GW frequency of 20 Hz) and the masses of the

two black holes.

3. How well can the eccentricity parameter be extracted from an observed event, using Bayesian parameter estimation and MCMC techniques? Are there any degeneracies between eccentricity and chirp mass, mass ratio, and/or spin that affect our ability to extract the eccentricity parameter from a data set?

III. MODELING AN ECCENTRIC BINARY BLACK HOLE INSPIRAL

A. Orbital Period and Eccentricity Decay

In his paper "Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries," Luc Blanchet derives the following coupled lowest order ordinary differential equations for period decay and eccentricity decay of BBH [2]:

$$\dot{P}_{orb} = -\frac{192\pi}{5c^5}\left(\frac{2\pi G}{P_{orb}}\right)^{5/3}\frac{m_1 m_2}{(m_1+m_2)^{1/3}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)\left(1-e^2\right)^{-7/2} \tag{1}$$

$$\dot{e} = -\frac{608\pi}{15c^5}\frac{e}{P_{orb}}\frac{m_1 m_2}{(m_1+m_2)^{1/3}}\left(1+\frac{121}{304}e^2\right)\left(1-e^2\right)^{-5/2} \tag{2}$$

It is important to note that these equations ignore spin effects and higher order effects. Furthermore, these equations only model the inspiral phase of the compact binary coalescence. Blanchet integrates analytically to determine that orbital period and eccentricity are related via

$$c_0 P^{19/9} = \frac{e^2}{(1-e^2)^{19/6}}\left(1+\frac{121}{304}e^2\right)^{145/121} \tag{3}$$

where c_0 is a constant determined by the initial conditions of the orbit. Numerically solving (1) and (2) in Python with the initial conditions of $P_{orb,0} = 0.3s$ and $e_0 = 0.4$ yields the following time series for orbital period and eccentricity (Figures 4 and 5):

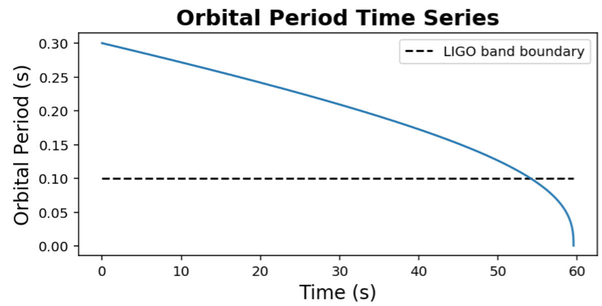

FIG. 4. Time evolution of orbital period for BBH system with initial orbital period of 0.3 seconds and initial eccentricity of 0.4

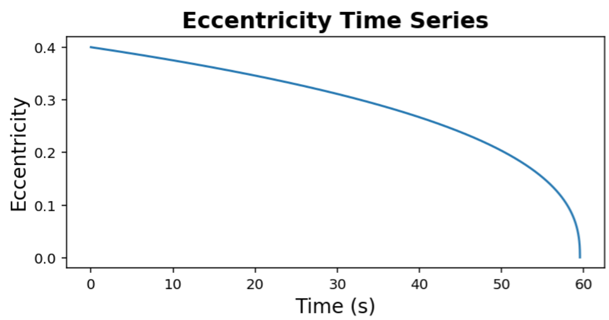

FIG. 5. Time evolution of eccentricity for BBH system with initial orbital period of 0.3 seconds and initial eccentricity of 0.4

These time series were generated from the given initial period to a final period corresponding the innermost stable circular orbit (ISCO), which is calculated in (4) and (5).

$$a_{ISCO} = 6\,\frac{G\,(\,m_1\,+\,m_2\,)}{c^2} \tag{4}$$

Combining (4) with Kepler's Third Law, given in (6), we get the period at ISCO:

$$P_{ISCO} = \sqrt{864}\frac{\pi\,G\,(\,m_1\,+\,m_2\,)}{c^3} \tag{5}$$

A time series for semi-major axis length a was calculated from Kepler's Third Law, seen in Figure 6:

$$a = \left[{P_{orb}}^2\left(\frac{G(m_1+m_2)}{4\pi^2}\right)\right]^{1/3} \tag{6}$$

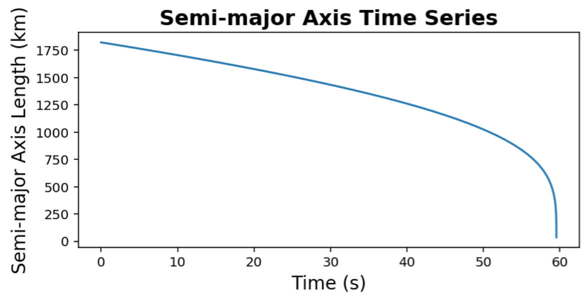

FIG. 6. Time evolution of semi-major axis length for BBH system with initial orbital period of 0.3 seconds and initial eccentricity of 0.4

Numerically Blanchet's equations in Python yields results that agree excellently with his analytic solution, as can be seen in Figure 7.

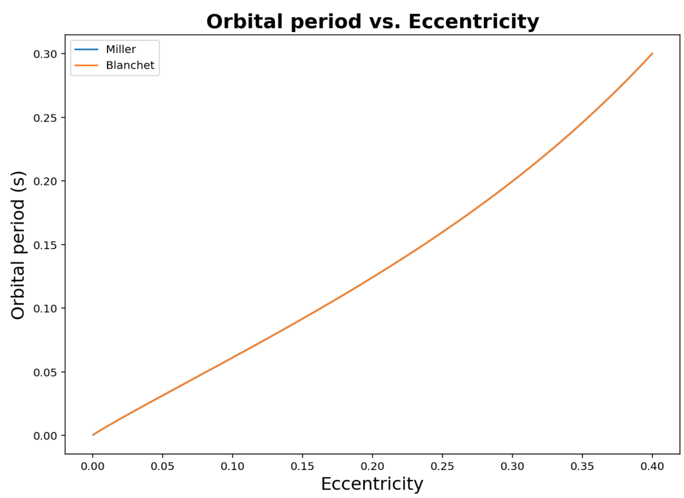

FIG. 7. Results from numerically solving Blanchet's equations compared with his analytic result (equation 3). The two lines cannot be distinguished - a perfect match.

The shape of the time series' for orbital period and eccentricity depend on initial orbital period, initial eccentricity, chirp mass, and mass ratio. As initial eccentricity, or for our purposes, the eccentricity at an orbital period of 0.1s (corresponding to an orbital frequency of 10 Hz, or a gravitational wave frequency of 20 Hz, the lowest frequency in the LIGO band) increases, the duration of the CBC in the LIGO band decreases; the BBH reaches ISCO in a shorter amount of time. As total mass increases and/or mass ratio decreases, the duration of the BBH in the LIGO band also decreases. These patterns can be seen in Figures 8, 9, and 10.

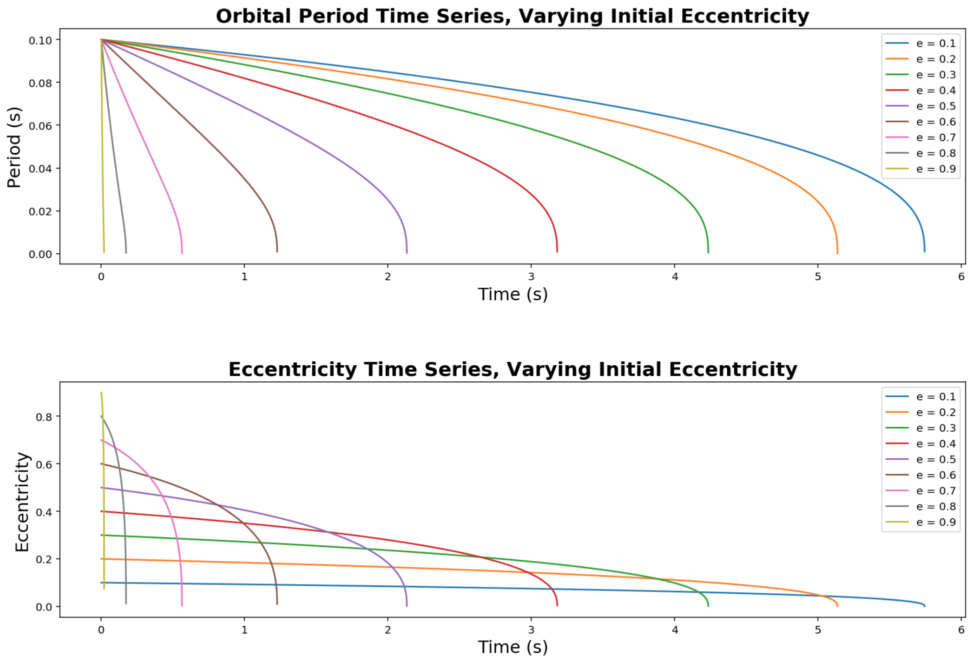

FIG. 8. Orbital period time series and eccentricity time series with varying values of eccentricity at an initial period of 0.1s.

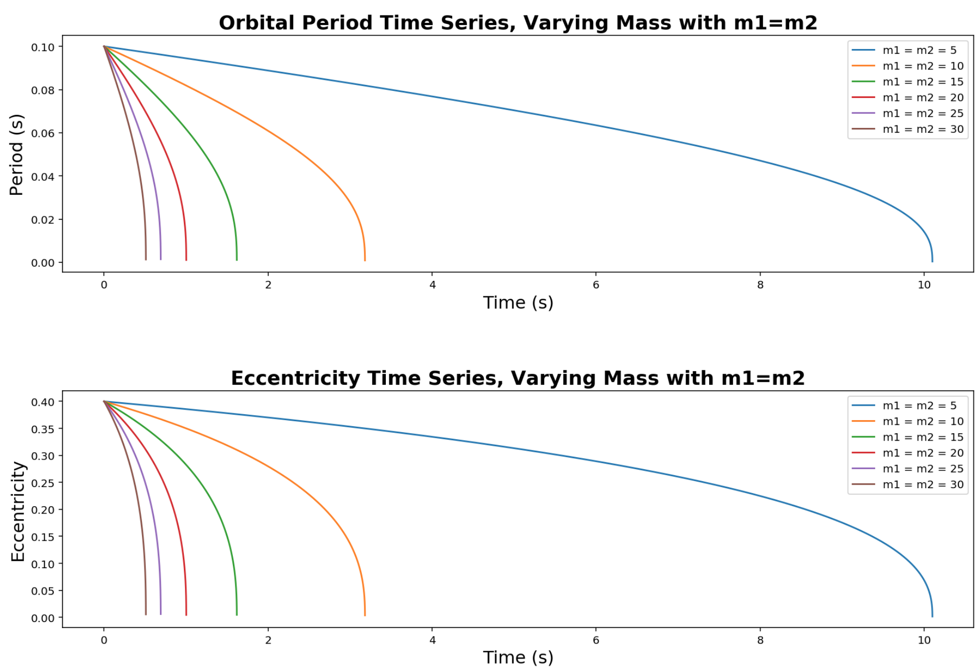

FIG. 9. Orbital period time series and eccentricity time series with varying total masses. All series have a mass ratio of 1.

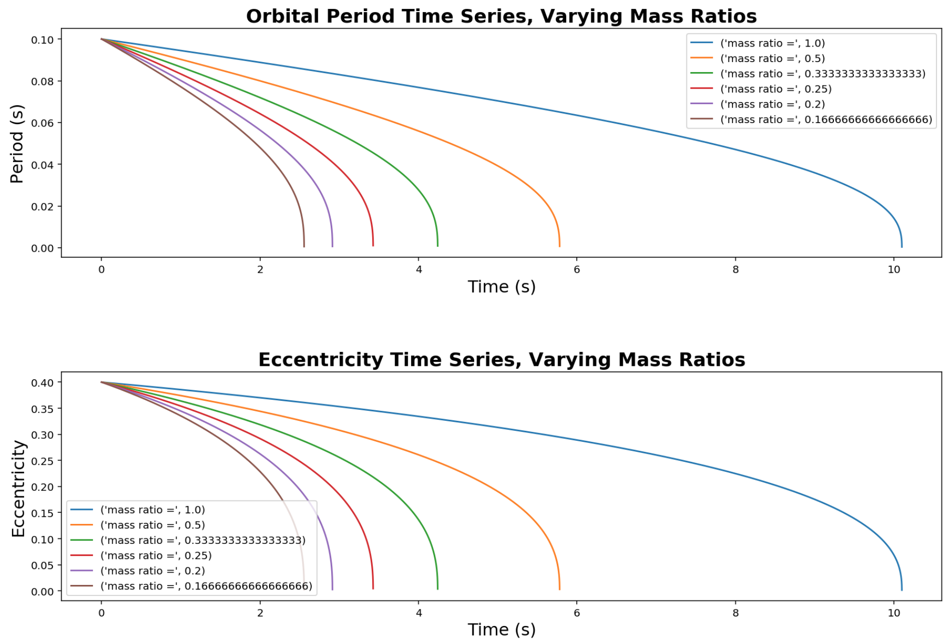

FIG. 10. Orbital period time series and eccentricity time series with varying mass ratios. m_1 always equals 5 solar masses, and m_2 ranges from 5 solar masses to 35 solar masses in increments of 5 solar masses.

B. Generating Waveform

Gravitational wave strain, $h(t)$ is generated by an accelerating quadrupole moment, I: $h(t) \sim \frac{d^2}{dt^2}(I)$ where $I = \int \rho\, r^2\, dV$. Without taking the effects of eccentricity into account, this strain is optimized in (7) where d is the distance to the source, a is the distance between the orbiting bodies, m_1 and m_2 are the masses of the BH, and $\varphi(t)$ is the phase evolution.

$$h(t) = \left(\frac{2G(m_1+m_2)}{c^2 d}\right)\left(\frac{2G(m_1+m_2)}{c^2 a}\right) cos(2\,\varphi(t)) \tag{7}$$

However, BBH in eccentric orbits produce a more complicated GW strain, due to emitting GW bursts at periastron passage, periastron precession, and the consequent oscillating distance between the orbiting bodies. The location of a body in an eccentric orbit can be defined using two angles: ψ and ϕ. ψ, also know as the true anomaly, corresponds to the radial period; it is taken with respect to the semi-major axis. However, due to periastron precession, the semi-major axis is itself rotating. ϕ takes this into account; it corresponds to the orbital period. It is taken from a fixed axis in spacetime, while the axis from which

ψ is taken is rotating. See Figure 11 for a visualization.

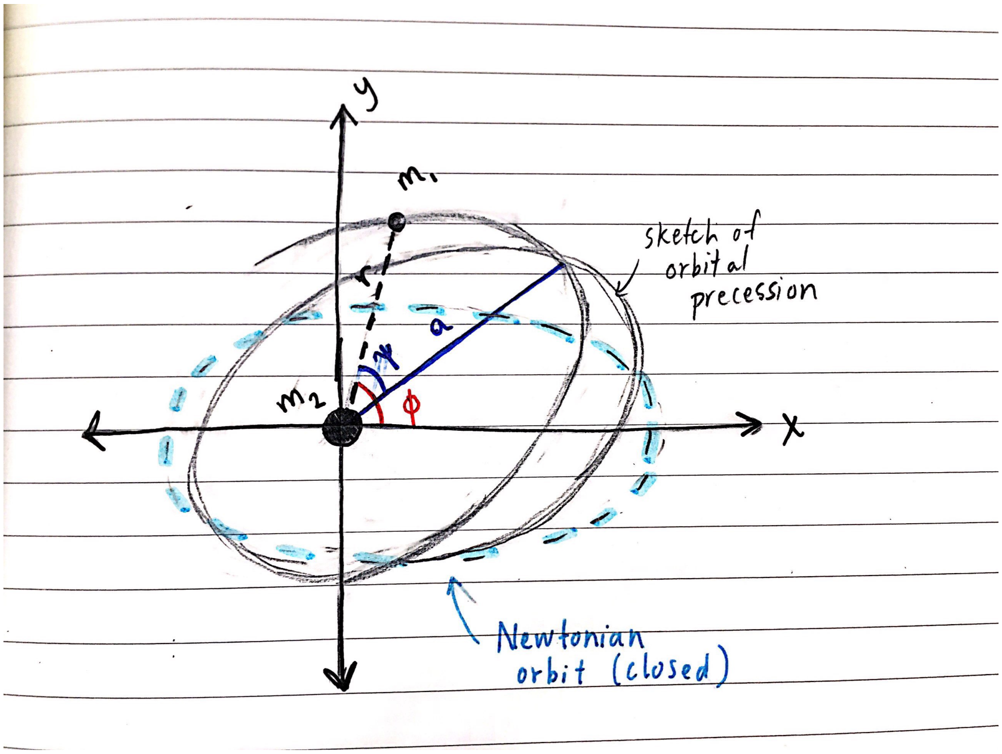

FIG. 11. Diagram showing difference between ϕ and ψ in an eccentric binary. ϕ is the angle between r and a fixed x-axis in m_2's (the larger mass) frame of reference, while ψ is the angle between r and the precessing semi-major axis a.

From [3], the plus and cross polarizations for gravitational wave strain from an eccentric BBH system are:

$$\begin{aligned} h_+(t) = \frac{\mu}{2D}\Bigg(&\Big[1 - 2\,cos(2\theta)\,cos^2(\phi(t)) - 3\,cos(2\,\phi(t))\Big]\dot{r}^2(t) \\ &+ \Big(3 + cos(2\theta)\Big)\Big[2\,cos(2\,\phi(t))\,\dot{\phi}^2(t) + sin(2\,\phi(t))\,\ddot{\phi}(t)\Big]r^2(t) \\ &+ \Big[4\Big(3 + cos(2\theta)\Big)\,sin(2\,\phi(t))\dot{\phi}(t)\dot{r}(t) \\ &+ \Big(1 - 2\,cos(2\theta)\,cos^2(\phi(t)) - 3\,cos(2\phi(t))\Big)\ddot{r}(t)\Big]r(t)\Bigg) \end{aligned} \quad (8)$$

$$h_\times(t) = -\frac{2\,\mu\,cos(\theta)}{D}\Bigg(sin\big(2\phi(t)\big)\,\dot{r}^2(t) + \Big[cos\big(2\phi(t)\big)\,\ddot{\phi}(t) \;-\; 2\,sin\big(2\phi(t)\big)\,\dot{\phi}^2(t)\Big]r^2(t)$$

$$+\Big[4\,cos\big(2\phi(t)\big)\,\dot{\phi}(t)\,\dot{r}(t) \;+\; sin\big(2\phi(t)\big)\,\ddot{r}(t)\Big]r(t)\Bigg) \quad (9)$$

where μ is the reduced mass of the binary, D is the distance to the source, θ is the angle of inclination of the source, r is the distance between the two BHs, and ϕ is angle corresponding to the orbital period. To solve for the strain, we need to time evolve r and ϕ which is done using the energy and angular momentum of the system. The Newtonian Mechanical definition of orbital energy E_{orb} for an elliptical orbit is (in SI units):

$$E_{orb} = K + U = \frac{U}{2} = -\frac{Gm_1m_2}{2a} \quad (10)$$

and the specific total energy of the system, in geometric units (to be used in equations throughout this report) is

$$E = \frac{E_{orb}}{\mu} + 1 \quad (11)$$

The angular momentum L of the system is is (in SI units, with $m_1 >= m_2$):

$$L = m_1v_1r_1 = m_2v_2r_2 = \big(2\,G\,m_1\,{m_2}^2\big)^{1/2}\left(\frac{1}{a(1+e)} + \frac{1}{a(1-e)}\right)^{-1/2} \quad (12)$$

where a is the semi-major axis length and e is the eccentricity. The distance r between the two BH's is:

$$r = \frac{a\left(1-e^2\right)}{1+e\,cos\psi} \quad (13)$$

where ψ is the true anomaly of the eccentric system (see Figure 11). To generate a time series for r, we must time evolve ψ using the following equation, adapted from [3]:

$$\dot{\psi} = \frac{(1\;-\;E^2)^{1/2}}{V_t\,(1\;-\;e^2)}\Big[a\,(1-e^2)\;-C_0\,(1-e)\;-\;e\,a\,(1-e^2) - e\,C_0\,(1-e)\,cos(\psi)\Big]^{1/2}$$

$$\Big[a\,(1-e^2)\,(1+e)\Big]^{1/2} \quad (14)$$

with the constant C_0 given by:

$$C_0 = \frac{2}{1 - E^2} - 2a \tag{15}$$

and potential V_t given by:

$$V_t = \frac{E\,r^4}{r^2\;-\;2r} \tag{16}$$

Finally, to solve the gravitational wave strain equations given in (8) and (9), we must generate a time series for ϕ, the angle describing where a body is in its orbital period. This is achieved with the following equation relating $\dot{\phi}$ to angular momentum, reduced mass, specific energy, angle of inclination, and distance between the BHs, also adapted from [3].

$$\dot{\phi} = \frac{L\,(1\;+\;E)\,r^3}{\mu\,sin\,^2\theta\,(r\;-\;2)} \tag{17}$$

Presently, I am working on debugging the methods in Python that I have written to numerically solve this system. I am generating time series for E, L, ψ, r, ϕ, h_+, and $h_\times$, and am comparing them to what these time series would look like for Newtonian elliptical orbits, i.e. orbits without the effects of energy loss via gravitational wave radiation.

IV. EXISTING WAVEFORMS IN LALSUITE

I addition to generating my own waveform for an eccentric BBH system, I have begun to explore the existing eccentric waveforms in LALSuite: EccentricTD in the time domain and EccentricFD in the frequency domain [5]. Both the waveforms in LALSuite and the waveform I am generating ignore spin effects and only model the inspiral phase of a CBC. Keeping all other parameters (masses, frequency range, etc.) constant, I plotted waveforms in the time and frequency domains at different initial eccentricities ranging from e=0.0001 to e=0.05. The results can be seen in Figures 12 and 13.

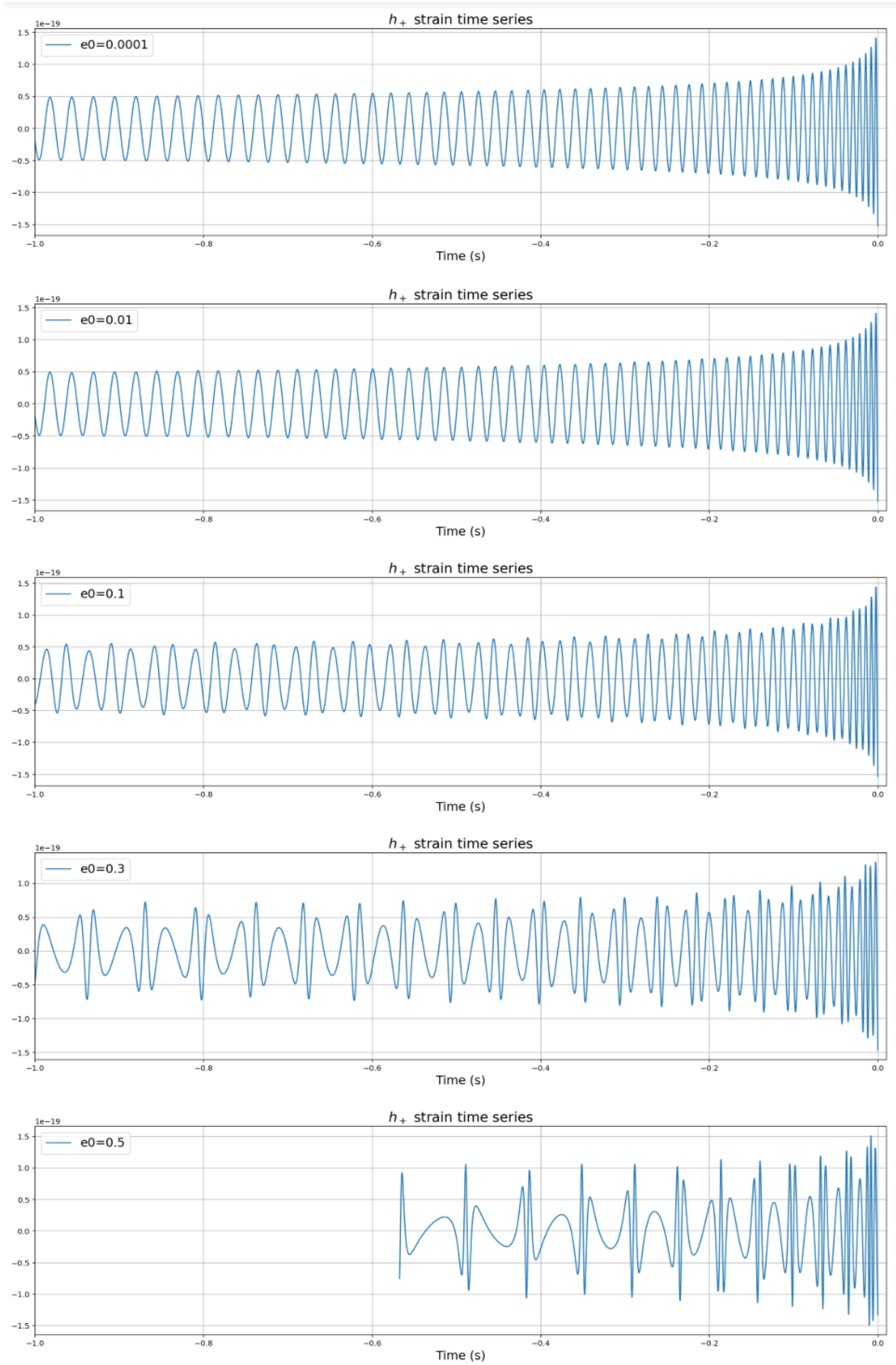

FIG. 12. Time series waveforms generated with the EccentricTD approximant at initial eccentricities e_0=[0.0001, 0.01, 0.1, 0.3, 0.5]. All waveforms shown in the final second before merger. Approximants taken from [5].

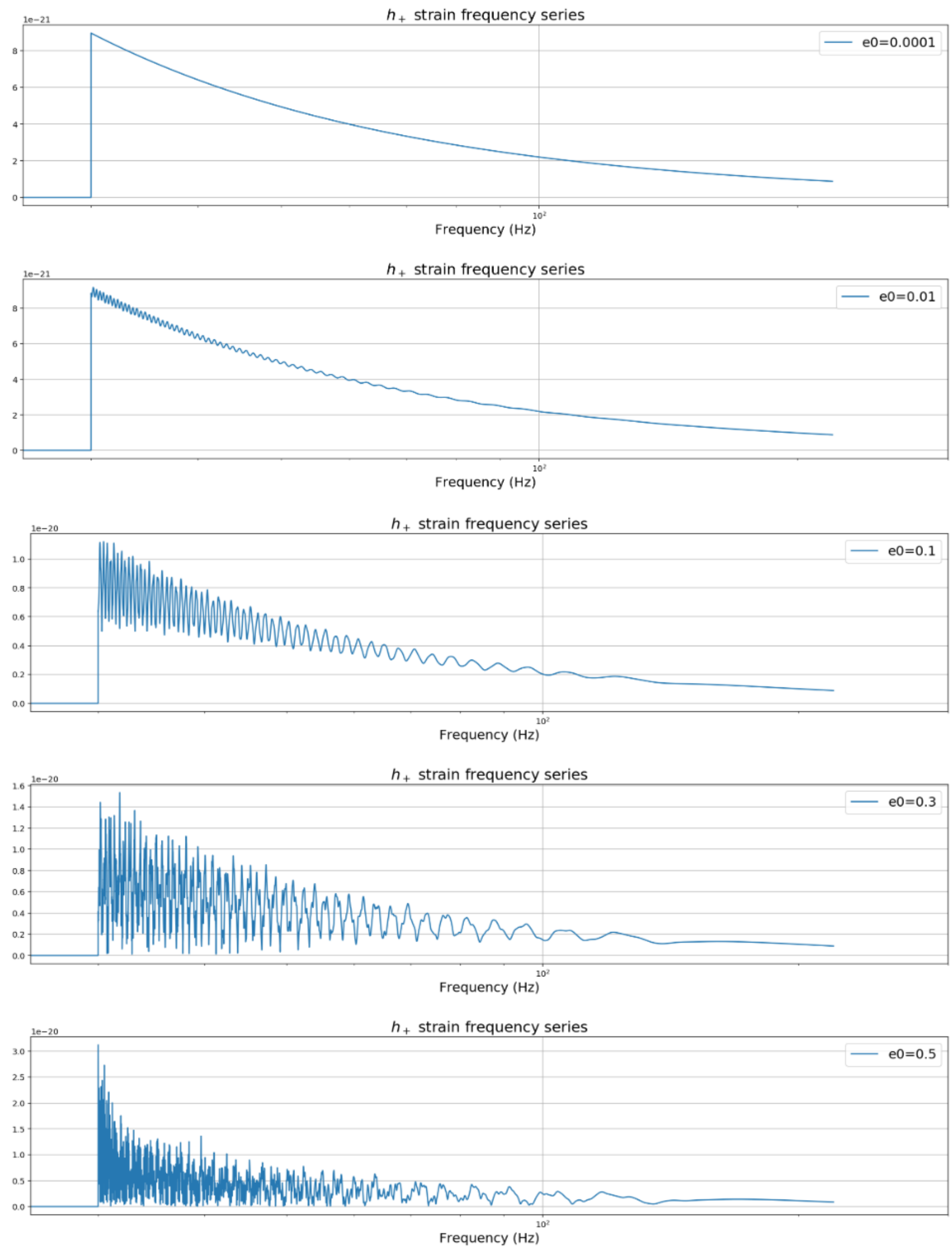

FIG. 13. Magnitude of frequency series waveforms generated with the EccentricFD approximant at initial eccentricities e_0=[0.0001, 0.01, 0.1, 0.3, 0.5]. The Frequency axis is on a logarithmic scale, in the range 25 Hz to 300 Hz. Approximants taken from [5].

As can be seen in the plots in Figure 12, as initial eccentricity in the LIGO band increases, the frequency of the strain has increasing periodic variation. This is a result of increasing amplitude of GW bursts at periastron passage. This is supported by the plots of the magnitude of the frequency series shown in Figure 13. When eccentricity is negligible, the frequency series is a smooth curve; as e_0 increases, we see more peaks in the frequency series, indicating that certain frequencies are more prominent than others due to the periodic GW bursts. Additionally, as e_0 increases, the BBH spends less time in the LIGO Band, reaching merger more quickly, for more energy is lost every periastron passage at higher eccentricities.

Finally, I have been working on defining a function to calculate phase accumulation of a waveform, which will be used show how two waveforms diverge in phase in the time or frequency domain, as well as a function to calculate the overlap between two waveforms. I will first apply these to non-eccentric templates with varying masses, such as the LALSuite approximant SEOBNRv4, to test their functionality, and will then apply them to EccentricTD and EccentricFD with varying values of e_0.

V. CHALLENGES

The challenges I have faced in the first three weeks of my LIGO SURF project have primarily been in debugging my code in Python. To generate my eccentric waveform, I have pulled equations from several different sources, each using different unit systems (SI, geometric, normalized by total mass, etc.) and have run into errors with correctly converting between different unit systems in Python. Additionally, I have found the models I am using to generate my waveform have often been too simple, and I have had to incorporate increasingly more parameters. For example, I began using only equation (7), in which eccentricity's affects on GW strain are over-simplified because orbital shape is not accounted for.

I expect to encounter other challenges similar to these throughout the summer. I also expect switching gears to focusing on data analysis techniques will be challenging, because of the learning curve for techniques such as Bayesian Inference and MCMC, which I have never utilized prior to this summer.

VI. FUTURE

From this point onwards, I will switch focus from generating a waveform to comparing this waveform with the existing "EccentricTD" and "EccentricFD" approximants in LALSuite, as well as using "EccentricTD" and "EccentricFD" waveforms for statistical analysis. My project schedule for the remainder of the summer is as follows:

Weeks 4-5: Finish debugging my eccentric waveform. If time, animate the orbit of a BBH system using the time series I derived for $r(t)$ and $\phi(t)$. Define and test functions to calculate phase accumulation for a single waveform, phases accumulation difference between two waveforms, overlap, likelihood, and posterior probability density. Apply these functions to EccentricTD and EccentricFD in LALSuite with different values of eccentricity, and use to compare eccentric waveforms to quasi-circular waveforms.

Week 6: Calculate how much SNR is lost we use quasi-circular binary templates to capture an eccentric binary merger event. We will calculate this as a function of the initial eccentricity (at GW frequency of 20 Hz) and the masses of the two black holes.

Week 7: Write second interim report. Using Bayesian inference techniques, begin searching for any degeneracies between eccentricity and chirp mass, mass ratio, and/or spin that could affect our ability to extract the eccentricity parameter from a dataset.

Week 8: Continue work from week 7. Generate corner plots for 1. chirp mass, mass ratio and eccentricity and 2. the results from an waveform with eccentricity and no spin vs. with spin and no eccentricity. Determine which existing quasi-circular waveform best matches a signal with eccentricity.

Week 9: Look at power distribution of eccentric waveforms vs. quasi circular waveforms using spectrograms (approach of the Burst Search group). Work on final presentation.

Week 10: Tie up loose ends. Finalize final presentation.

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[1] Abbott, B., et al. (LIGO Scientific Collaboration), "Astrophysical Implications of the Binary Black-Hole Merger GW150914," *Astrophys. J.*, LIGO-P1500262, (2016), [arXiv:1602.03846v1].

[2] Blanchet, Luc. ”Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries,” *Living Rev. Relativity*, 17 /lrr-2014-2 (2014):http://www.livingreviews.org/lrr-20014-2.

[3] Callister, Thomas A. ”Improving Models of Eccentric Compact Binary Inspirals,” University of Cambridge, (2014).

[4] East, William B., et al. ”Observing complete gravitational wave signals from dynamical capture binaries”, *Phys. Rev. D*, 87 (2013), [arXiv:1212.0837v2]

[5] LALSuite: Main Page, https://lscsoft.docs.ligo.org/lalsuite/

[6] LIGO Open Science Center, Data release for event GW150914, https://losc.ligo.org/events/GW150914/

[7] Lower, Marcus E., et al. ”Measuring eccentricity in binary black hole inspirals with gravitational waves,” (2018)

[8] Randall, Lisa and Zhong-Zhi Xianyu. ”Induced Ellipticity for Inspiraling Binary Systems,” (2018), [arXiv:1708.08569v2]

[9] Randall, Lisa and Zhong-Zhi Xianyu. ”Induced Ellipticity for Inspiraling Binary Systems,”An Analytical Portrait of Binary Mergers inHierarchical Triple Systems,” (2018) [arXiv:1802.05718v1]

[10] Samsing, Johan. ”Eccentric Black Hole Mergers Forming in Stellar Clusters,” (2017), [arXiv:1711.07452]

[11] Sathyaprakasch, B.S and Bernard F. Schutz, Physics, Astrophysics, and Cosmology with Gravitational Waves, *Living Rev. Relativity*, lrr-2009-2, (2009): http://www.livingreviews.org/lrr-2009-2.

[12] Tiwari, V., et al. ”A Proposed Search for the Detection of Gravitational Waves from Eccentric Binary Black Holes,” *Phys. Rev. D*, 93 (2016), [arXiv:1511.09240]