Relative Intensity Noise in Reflection of an Optical Resonator

Theory

We are looking at the relative intensity noise (RIN) in reflection of an optical resonator. In particular, we are looking at the RIN above the cavity pole, where intensity fluctuations are promptly reflected. The incoming laser beam is phase modulated by 2 RF frequencies to support the Pound-Drever-Hall reflection locking technique as well as the Michelson sensing using the Schnupp asymmetry. We write in incoming E-field:

$$E_{\text{in}} = \left(\sqrt{1 - \frac{\Gamma_1^2 + \Gamma_2^2}{2}} + i\,\Gamma_1\,\text{Cos}[\omega_1 t] + i\,\Gamma_2\,\text{Cos}[\omega_2 t]\right)\left(1 + \frac{\Delta A}{A}\,\frac{\text{Cos}[\omega_a t]}{\sqrt{2}}\right) E_0$$

where Γ_1 and Γ_2 are the RF modulation index of the 2 RF sidebands, respectively, $E_o = \sqrt{P_0}$ is the field amplitude representing the input power P_0, ω_1 and ω_2 are the RF modulation frequencies, ω_a is the frequency of the amplitude modulation, and $\Delta A/A$ represents the RIN of the incoming beam.
We can calculate the intensity noise using the following Ansatz:

$$\frac{P_{\text{in}}}{P_0} = \frac{\overline{E_{\text{in}}\,E_{\text{in}}^*}}{P_0} = 1 + \frac{\sqrt{2}\;\Delta A\,\text{Cos}[t\,\omega_a]}{A}$$

where the bar indicates an average over all RF sidebands powers. The RMS of the second term is $\Delta A/A$ and yields the RIN.

The cavity length maybe chosen so that the RF frequencies are resonant or not. For the LIGO interferometer ω_1 and ω_2 are 9.1 and 45.5 MHz and are both resonant in the power recycling cavity. We now write the amplitude reflection coefficients for a TEM_{00} resonant mode as r_0, r_1, and r_2 for carrier and the 2 RF sidebands respectively. When the cavity is over-coupled this coefficient is negative. The amplitude reflection coefficient for intensity noise fluctuations above the cavity pole is $r_a = 1$, meaning these fluctuations are promptly reflected.

We know can decompose the incoming E-field into TEM_{00} and higher order modes TEM_{HOM}, where we lump all higher order modes into one field without loosing any generality, i.e.,

$$E = \text{TEM}_{00} + \text{TEM}_{\text{HOM}}$$

where we normalize

$$\overline{\text{TEM}_{00}\,\text{TEM}_{00}^*} = (1 - m^2)\,|E_0|^2,\quad \overline{\text{TEM}_{\text{HOM}}\,\text{TEM}_{\text{HOM}}^*} = m^2\,|E_0|^2 \text{ and } \overline{\text{TEM}_{00}\,\text{TEM}_{\text{HOM}}^*} = 0$$

and where we use the bar to represent an integration over the transverse degrees-of-freedom.

We note that the amplitude reflection coefficients for these higher order modes are all $r_{HOM} = 1$. We can now write the E-field in reflection of the cavity as

$$E_{refl} = \left(r_0 \sqrt{1 - \frac{\Gamma_1^2 + \Gamma_2^2}{2}} + i\, r_1\, \Gamma_1\, \text{Cos}[\omega_1\, t] + i\, r_2\, \Gamma_2\, \text{Cos}[\omega_2\, t]\right) \text{TEM}_{00} +$$
$$\left(\sqrt{1 - \frac{\Gamma_1^2 + \Gamma_2^2}{2}} + i\, \Gamma_1\, \text{Cos}[\omega_1\, t] + i\, \Gamma_2\, \text{Cos}[\omega_2\, t]\right) \frac{\Delta A}{A} \frac{\text{Cos}[\omega_a\, t]}{\sqrt{2}}\, \text{TEM}_{00} +$$
$$\left(\sqrt{1 - \frac{\Gamma_1^2 + \Gamma_2^2}{2}} + i\, \Gamma_1\, \text{Cos}[\omega_1\, t] + i\, \Gamma_2\, \text{Cos}[\omega_2\, t]\right)\left(1 + \frac{\Delta A}{A} \frac{\text{Cos}[\omega_a\, t]}{\sqrt{2}}\right) \text{TEM}_{HOM}$$

where the first term is the TEM_{00} term that enters the resonator, the second term is the prompt reflected amplitude modulated TEM_{00} field, and the third term is the prompt reflected higher order mode term TEM_{HOM}. The total reflected power at DC then becomes:

$$\frac{P_{refl}}{P_0} = \frac{\overline{E_{refl}\, E_{refl}^*}}{P_0} = (1 - m^2)\left(r_1^2 \frac{\Gamma_1^2}{2} - r_2^2 \frac{\Gamma_2^2}{2} + r_0^2\left(1 - \frac{\Gamma_1^2 + \Gamma_2^2}{2}\right)\right) + m^2 +$$
$$\frac{\sqrt{2}\ \Delta A\, \text{Cos}[t\, \omega_a]}{A}\left((1 - m^2)\left(r_1 \frac{\Gamma_1^2}{2} + r_2 \frac{\Gamma_2^2}{2} + r_0\left(1 - \frac{\Gamma_1^2 + \Gamma_2^2}{2}\right)\right) + m^2\right)$$

The ratio of the RIN in reflection over the input then becomes

$$\frac{\text{RIN}_{refl}}{\text{RIN}_{in}} == \frac{(1 - m^2)\left(r_1 \frac{\Gamma_1^2}{2} + r_2 \frac{\Gamma_2^2}{2} + r_0\left(1 - \frac{\Gamma_1^2 + \Gamma_2^2}{2}\right)\right) + m^2}{(1 - m^2)\left(r_1^2 \frac{\Gamma_1^2}{2} - r_2^2 \frac{\Gamma_2^2}{2} + r_0^2\left(1 - \frac{\Gamma_1^2 + \Gamma_2^2}{2}\right)\right) + m^2}$$

If the modulation indices are small, we get the approximation

$$\frac{\text{RIN}_{refl}}{\text{RIN}_{in}} \approx \frac{(1 - m^2)\, r_0 + m^2}{(1 - m^2)\, r_0^2 + m^2} \approx \frac{r_0 + m^2}{r_0^2 + m^2}$$

where for the last approximation we also assumed that $m^2 \ll 1$.

Setup

```
<< Notation`
Symbolize[ ΔA ]
```

In[1]:=
```
EEconj[e_] := PowerExpand[Simplify[ComplexExpand[e e*] / E0^2, 1 - Γ1^2 - Γ2^2 > 0 ∧ 1 - m^2 > 0]]
RFavrg[f_] :=
 Normal[Series[Integrate[Integrate[f, {ω1, 0, 2π/t}], {ω2, 0, 2π/t}] (t/(2π))^2, {ΔA, 0, 1}]]
```

Scratch

Old

Equations

RIN and Reflected Power Ratio

R: Carrier reflectivity coefficient (power)
M: Mode mismatch coefficient (power)
sign: Sign of carrier amplitude reflectivity (negative for over-coupled)

In[3]:=
```
reflrin[R_, M_, sign_ : -1] := (M + sign Sqrt[R] (1 - M))/(M + R (1 - M))

reflR[R_, M_] := M + R (1 - M)
```

Solutions for Mode Match and Carrier Reflectivity

Solve for R and M given a measured RIN ratio and a measured reflected power ratio

In[5]:=
```
reflSol[RRIN_, Rifo_, sign_ : -1] :=
 Solve[{(M + sign Sqrt[R] (1 - M))/(M + R (1 - M)) == RRIN, M + R (1 - M) == Rifo}, {R, M}]

reflSolSimple[RRIN_, Rifo_, sign_ : -1] := Solve[{(M + sign Sqrt[R])/(M + R) == RRIN, M + R == Rifo}, {R, M}]
```

Examples

75 W Input

In[7]:=
```
reflSol[-2.305, 0.0732]
reflSolSimple[-2.305, 0.0732]
```

Out[7]= {{R → 0.0428489, M → 0.0317098}}

Out[8]= {{R → 0.0405516, M → 0.0326484}}

60 W Input

In[9]:=
```
reflSol[-2.75, 0.0686]
reflSolSimple[-2.75, 0.0686]
```

Out[9]=
```
{{R → 0.0468385, M → 0.0228309}}
```

Out[10]=
```
{{R → 0.0450351, M → 0.0235649}}
```

In[11]:=
```
reflSol[-2.75, 0.9 × 0.0686]
reflSolSimple[-2.75, 0.9 × 0.0686]
```

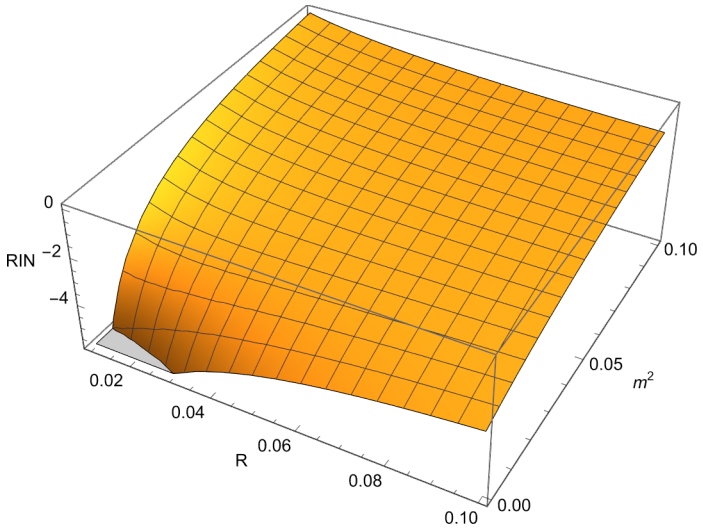
Out[11]=
```
{{R → 0.0391727, M → 0.0234873}}
```

Out[12]=
```
{{R → 0.037605, M → 0.024135}}
```

Plots

In[•]:=
```
Plot3D[reflrin[R, M], {R, 0.01, 0.1}, {M, 0.00, 0.1}, AxesLabel → {"R", "m^2", "RIN"}]
```

Out[•]=



In[•]:=
```
ContourPlot[reflrin[R, M], {R, 0.0, 0.2}, {M, 0.00, 0.1}, PlotLegends → Automatic,
 AspectRatio → 0.5, PlotRange → {-8, 1}, Contours → Range[-8, 1, 0.5],
 FrameLabel → {"Interferometer Reflectivity (R)", "Mode Mismatch (m^2)"}, ImageSize → Large]
```

Out[•]=